Day 213 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 34 cm and its area is 70 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 34$ and $\frac{1}{2}r^2\theta = 70$

$r\theta = 34 - 2r$, so $\frac{1}{2}r(34 - 2r) = 70$

$r^2 - 17r + 70 = 0$, so $(r - 7)(r - 10) = 0$

$r = 7$: $r\theta = 20$, $\theta = \frac{20}{7}$ $r = 10$: $r\theta = 14$, $\theta = \frac{7}{5}$

Answer: r = 7, $\theta = \frac{20}{7}$ or r = 10, $\theta = \frac{7}{5}$

2 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:
$4^x \times 16^y = 1024$
$3^x \div 9^y = 3$

Write each side with prime bases: $1024 = 2^{10}$, $3 = 3^1$.
First: $2^{2x} \times 2^{4y} = 2^{10}$, so $2x + 4y = 10$.
Second: $3^{1x - 2y} = 3^1$, so $1x - 2y = 1$.
Solving the pair of linear equations: x = 3, y = 1.
Answer: x = 3, y = 1

3 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A seed supplier says that 10% of its seeds germinate. 15 seeds are planted, and 0 germinate.
Test, at the 5% significance level, the claim that the proportion is lower.
Give your conclusion in context.

H_0: p = 0.1, H_1: p < 0.1. Let X ~ B(15, 0.1)
$P(X \leq 0) = 0.206$
0.206 is greater than 5%, so do not reject H_0
There is insufficient evidence that the proportion is lower than the claimed 0.1
Answer: Do not reject H_0: p-value 0.206 > 5%

4 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 30\,e^{-x} = 11$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 11e^x + 30 = 0$
$(e^x - 6)(e^x - 5) = 0$
$e^x = 5$ or $e^x = 6$
Answer: x = ln 5 and x = ln 6

5 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 6 kg and 4 kg hang on a light string over a smooth fixed pulley. The 6 kg particle is 0.75 m above the ground; the 4 kg one is on the ground.
Released from rest, find (a) the tension, (b) the greatest height reached by the 4 kg particle. Take g = 9.8 m s^{-2}.

$6g - T = 6a$ and $T - 4g = 4a$, so $a = \frac{6 - 4}{6 + 4} \times 9.8 = 1.96$

$T = 6(9.8 - 1.96) = 47.04$

The 4 kg particle rises 0.75 m before the 6 kg particle lands, reaching $v^2 = 2 \times 1.96 \times 0.75 = 2.94$

The string goes slack, so it then moves freely under gravity: extra height = $\frac{v^2}{2g} = \frac{2.94}{19.6} = 0.15$

Greatest height = 0.75 + 0.15
Answer: (a) 47.04 N (b) 0.9 m

6 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 3x^2 + 11x + 18$.

Show that $(x + 2)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-2) = -8 + 12 - 22 + 18 = 0$, so $(x + 2)$ is a factor.

$f(x) = (x + 2)(x^2 + x + 9)$.

For $x^2 + x + 9 = 0$: $b^2 - 4ac = 1 - 36 = -35 < 0$.

No real roots from the quadratic, so the only real root is $x = -2$.

Answer: Proved; the only real root is $x = -2$

7 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(15, \frac{3}{10})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{15 - r}{r + 1} \times \frac{3}{7}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(15, r + 1)}{C(15, r)} \times \frac{p}{q} = \frac{15 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{3}{10}$ and $q = \frac{7}{10}$, $p \div q = \frac{3}{7}$

The ratio exceeds 1 when $(15 - r)\frac{3}{10} > (r + 1)\frac{7}{10}$, that is $r < 3.8$

So probabilities rise up to $r = 4$ and then fall; the mode is 4

$P(X = 4) = 0.219$

Answer: 4 with probability 0.219

8 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_1^3 x^2 \ln x \, dx$

Let $u = \ln x$ and $dv/dx = x^2$, so $du/dx = \frac{1}{x}$, $v = \frac{x^3}{3}$

Integral $= [\frac{x^3}{3} \ln x] - \int \frac{x^2}{3} \, dx$

$= [\frac{x^3}{3} \ln x - \frac{x^3}{9}]$

Answer: $-\frac{26}{9} + 9\ln 3$

9 Car on a hill: power and maximum speed (CIE)

A car of mass 800 kg, with engine power 9.6 kW, climbs a straight road with $\sin \alpha = \frac{1}{14}$ against a constant resistance of 400 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at 5 m s^{-1}. Take $g = 9.8$ m s^{-2}.

Steady speed: driving force $= 400 + 800 \times 9.8 \times \frac{1}{14} = 960$ N

$P = F v$, so $v = \frac{9600}{960} = 10$

At 5 m s^{-1}: $F = \frac{9600}{5} = 1920$ N

$F - 960 = 800 a$, so $a = \frac{960}{800}$

Answer: (a) 10 m s^{-1} (b) 1.2 m s^{-2}

10 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = 3$ and $y = -2$, so $p = -3$ and $q = -2$.

$y = \frac{a}{x - 3} - 2$.

P(2, −8): $-8 = \frac{a}{-1} - 2$, so $a = 6$.

Answer: a = 6, p = −3, q = −2