Day 212 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 5x^2 + 8x + 6$.
Show that $(x + 3)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-3) = -27 + 45 - 24 + 6 = 0$, so $(x + 3)$ is a factor.

$f(x) = (x + 3)(x^2 + 2x + 2)$.

For $x^2 + 2x + 2 = 0$: $b^2 - 4ac = 4 - 8 = -4 < 0$.

No real roots from the quadratic, so the only real root is $x = -3$.

Answer: Proved; the only real root is $x = -3$

2 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 20 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 95$, and after 5 minutes $\theta = 70$.
Find A and k, then find when $\theta = 40$ (3 s.f.).

$t = 0$: $20 + A = 95$ so $A = 75$

$t = 5$: $20 + 75e^{-5k} = 70$ so $e^{-5k} = \frac{2}{3}$

$k = \frac{1}{5} \ln \frac{3}{2} = 0.0811$

$75\, e^{-kt} = 20$ so $t = \frac{1}{k} \ln \frac{15}{4}$

$t = 16.3$ minutes

Answer: $A = 75$, $k = 0.0811$, $t = 16.3$ minutes

3 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 8 seats. Each row has 5 more seats than the row in front. Find
(a) the number of seats in row 32,
(b) the least number of rows for at least 626 seats.

Arithmetic: $a = 8$, $d = 5$.

(a) $u_{32} = 8 + 31 \times 5 = 163$.

(b) $S_n = \frac{n}{2}(16 + 5n - 5) = \frac{n(5n + 11)}{2} \geq 626$.

$S_{14} = 567 < 626$, $S_{15} = 645 \geq 626$.

Answer: (a) 163 (b) 15 rows

4 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At $t = 0$ seconds P is at $(-5\mathbf{i} - 2\mathbf{j})$ m with velocity $(-7\mathbf{i} + 14\mathbf{j})$ m s^{-1}. At $t = 10$, P has velocity $(33\mathbf{i} - 6\mathbf{j})$ m s^{-1}.
Find (a) the acceleration of P, (b) the time at which P moves parallel to $(3\mathbf{i} + 2\mathbf{j})$, (c) the position vector of P at that time.

$\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t} = \frac{(33\mathbf{i} - 6\mathbf{j}) - (-7\mathbf{i} + 14\mathbf{j})}{10} = (4\mathbf{i} - 2\mathbf{j})$

$\mathbf{v} = (-7\mathbf{i} + 14\mathbf{j}) + t(4\mathbf{i} - 2\mathbf{j})$

Parallel to $(3\mathbf{i} + 2\mathbf{j})$: $(4t - 7) \times 2 = (-2t + 14) \times 3$, giving $t = 4$

$\mathbf{r} = (-5\mathbf{i} - 2\mathbf{j}) + 4(-7\mathbf{i} + 14\mathbf{j}) + \frac{1}{2} \times 4^2 (4\mathbf{i} - 2\mathbf{j}) = (-\mathbf{i} + 38\mathbf{j})$

Answer: (a) $(4\mathbf{i} - 2\mathbf{j})$ m s^{-2} (b) $t = 4$ (c) $(-\mathbf{i} + 38\mathbf{j})$ m

5 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that

$$\frac{\sin 5\theta \tan 5\theta}{1 - \cos 4\theta}$$

approaches as $\theta \to 0$.

$\sin 5\theta \approx 5\theta$, $\tan 5\theta \approx 5\theta$, $1 - \cos 4\theta \approx 8\theta^2$

Expression $\approx \frac{5\theta \times 5\theta}{8\theta^2} = \frac{25}{8}$

Answer: $\frac{25}{8}$

6 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The sum of the first n terms of a sequence is

$S_n = 3n^2 + 9n$.

(a) Find an expression for u_n.

(b) Find the least n for which $u_n > 192$.

$u_n = S_n - S_{n-1} = 3n^2 + 9n - [3(n-1)^2 + 9(n-1)]$

$= 6n + 6$.

$u_1 = S_1 = 12$ also fits, so $u_n = 6n + 6$ for all n.

$6n + 6 > 192 \Rightarrow n > 31$, so $n = 32$.

Answer: (a) $u_n = 6n + 6$ (b) $n = 32$

7 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ab^x$. The line of best fit for lg y against x is $\lg y = 0.7 + 0.5x$.

(a) Find a and b to 3 significant figures.

(b) Estimate y when x = 16, to 3 significant figures.

(c) Interpret the value of b.

$y = ab^x$ gives $\lg y = \lg a + x \lg b$

$\lg a = 0.7$ so $a = 10^{0.7} = 5.012$

$\lg b = 0.5$ so $b = 10^{0.5} = 3.162$

$x = 16$: $\lg y = 0.7 + 0.5 \times 16 = 8.7$

$y = 10^{8.7} = 501000000$

b is the factor y is multiplied by each time x increases by 1 (growth of 220%)

Answer: (a) a = 5.01, b = 3.16 (b) 501000000 (c) y grows by a factor of 3.16 per unit of x

8 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve $y = \frac{18}{x}$ and the line $y = 9 - x$.

Find the exact area of the shaded region enclosed by the curve and the line.

Intersect: $\frac{18}{x} = 9 - x$ gives $x^2 - 9x + 18 = 0$, so $x = 3$ and $x = 6$

Area $= \int_3^6 (9 - x - \frac{18}{x})\,dx = [9x - \frac{x^2}{2} - 18 \ln x]$

$= (54 - 18 - 18 \ln 6) - (27 - \frac{9}{2} - 18 \ln 3)$

Answer: $\frac{27}{2} - 18\ln 2$

9 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(17, \frac{3}{10})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{17 - r}{r + 1} \times \frac{3}{7}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(17, r+1)}{C(17, r)} \times \frac{p}{q} = \frac{17 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{3}{10}$ and $q = \frac{7}{10}$, $p \div q = \frac{3}{7}$

The ratio exceeds 1 when $(17 - r)\frac{3}{10} > (r + 1)\frac{7}{10}$, that is $r < 4.4$

So probabilities rise up to r = 5 and then fall; the mode is 5

$P(X = 5) = 0.208$

Answer: 5 with probability 0.208

10 Rough slope: up then back down (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 5 m s^{-1} up a line of greatest slope of a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{4}{5}$ and $\mu = 0.95$.

Find (a) the distance it travels up the plane, to 3 s.f., (b) its speed on returning to its starting point. Take g = 9.8 m s^{-2}.

$\cos\alpha = \frac{3}{5}$. Up the plane: R = mg cos α, so $a_{up} = g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{4}{5} + 0.95 \times \frac{3}{5}) = 13.426$

$0 = 5^2 - 2 \times 13.426 \times d$, so $d = \frac{25}{26.852} = 0.9310$

Down the plane: $a_{down} = g(\sin\alpha - \mu\cos\alpha) = 9.8(\frac{4}{5} - 0.95 \times \frac{3}{5}) = 2.254$

$v^2 = 2 \times 2.254 \times 0.93103 = 4.197$, so v = 2.05

Answer: (a) 0.931 m (b) 2.05 m s^{-1}