Day 211 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -3$ and $y = 3$, so $p = 3$ and $q = 3$.

$y = \frac{a}{x + 3} + 3$.

P$(-2, 9)$: $9 = \frac{a}{1} + 3$, so $a = 6$.

Answer: a = 6, p = 3, q = 3

2 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (2, 16) and (4, 64). Find the values of a and n.

$16 = a \times 2^n$ and $64 = a \times 4^n$

Divide: $2^n = 4$ so $n = 2$

$a = 16 \div 2^2 = 4$

Answer: a = 4, n = 2

3 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-3t^2 + 9t)\mathbf{i} + (2.5t^2 - 6t)\mathbf{j}$.
Find (a) the value of t at which P is moving parallel to $(\mathbf{i} - \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-6t + 9)\mathbf{i} + (5t - 6)\mathbf{j}$

Parallel to $(\mathbf{i} - \mathbf{j})$: $(-6t + 9) \times -1 = (5t - 6) \times 1$

t(1) = 3, so t = 3

At t = 3: $\mathbf{v} = (-9\mathbf{i} + 9\mathbf{j})$, speed $= \sqrt{(-9^2 + 9^2)}$

Answer: (a) t = 3 (b) 12.7 m s^{-1}

4 Poisson: scaling the rate, then binomial (CIE)

Calls arrive at a switchboard at an average rate of 12 per hour, independently and at random.
(a) Find the probability of at least 2 in a 30-minute period.
(b) Find the probability that, of 7 separate 30-minute periods, exactly 2 have no arrivals.

30-minute period: $\lambda = 12 \times \frac{30}{60} = 6$, so $X \sim Po(6)$

(a) $P(X \geq 2) = 1 - P(X \leq 1) = 0.9826$

(b) $P(X = 0) = e^{-6} = 0.002479$

$Y \sim B(7, 0.002479)$: $P(Y = 2) = 0.000127$

Answer: (a) 0.983 (b) 0.000127

5 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, solve $\cos 3\theta = \cos\theta$ for $0 < \theta < 2\pi$, in terms of π.

Substitute: $\cos\theta(4\cos^2\theta - 3 - 1) = 0$

$\cos\theta = 0$ or $\cos^2\theta = 1$

$\cos\theta = 0$ gives $\frac{\pi}{2}, \frac{3\pi}{2}$

$\cos^2\theta = 1$ gives π

Answer: $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

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6 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(5\mathbf{i} + 47\mathbf{j})$ m s^{-1} and $(14\mathbf{i} + 35\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).
Both are still in flight after 3 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $5t\mathbf{i} + (47t - 4.9t^2)\mathbf{j}$; position of B: $14t\mathbf{i} + (35t - 4.9t^2)\mathbf{j}$
Subtracting, the $4.9t^2$ terms cancel: AB = $t(-9\mathbf{i} + 12\mathbf{j})$
After 3 s: AB = $(-27\mathbf{i} + 36\mathbf{j})$
Distance = $\sqrt{(27^2 + 36^2)} = 45$

The direction of AB is that of $(-9\mathbf{i} + 12\mathbf{j})$, which does not depend on t: tan angle = $\frac{12}{9}$

Answer: (a) 45 m (b) 53.1°

7 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x + 1)
the remainder is 3; when divided by (x − 3)
the remainder is 23. Find the remainder when
f(x) is divided by (x + 1)(x − 3).

f(x) = (x + 1)(x − 3)Q(x) + px + q.
f(−1) = 3: −p + q = 3.
f(3) = 23: 3p + q = 23.
Subtracting: −4p = −20, so p = 5, q = 8.
Answer: 5x + 8

8 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 2 and common ratio 4.
Find the least n for which $S_n > 18371$.

$S_n = \frac{2(4^n - 1)}{3} > 18371$.

$4^n > 27557.50$.
$n > \frac{\log(2.756e+04)}{\log(4)} = 7.375$.
So the least n is 8.
Answer: n = 8

9 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$.
The point P has coordinates (5, 7, −7).
Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 3 \\ 11 \\ -7 \end{pmatrix} - t\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$
FP · d = 0: 30 − t(10) = 0, so t = 3
F = (−1, 5, 0)
$\mathrm{PF} = \begin{pmatrix} 6 \\ 2 \\ -7 \end{pmatrix}$, so PF = $\sqrt{89}$

Answer: F = (−1, 5, 0), PF = $\sqrt{89}$

10 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(120, 0.35).

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate P(X ≥ 43), to 3 significant figures.

np = 42 and nq = 78, both greater than 5 (n large, p not extreme)

Y ~ N(42, 27.3) with σ = 5.225

Apply the continuity correction:

P(X ≥ 43) ≈ P(Y > 42.5) = P(Z > 0.096) = 0.462

Answer: 0.462