Day 210 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 20 cm and its area is 21 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 20$ and $\frac{1}{2}r^2\theta = 21$

$r\theta = 20 - 2r$, so $\frac{1}{2}r(20 - 2r) = 21$

$r^2 - 10r + 21 = 0$, so $(r - 3)(r - 7) = 0$

$r = 3$: $r\theta = 14$, $\theta = \frac{14}{3}$ $r = 7$: $r\theta = 6$, $\theta = \frac{6}{7}$

Answer: $r = 3, \theta = \frac{14}{3}$ or $r = 7, \theta = \frac{6}{7}$

2 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 - 6x + 6y - 151 = 0$
passes through P(−2, 9). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (3, −3). Gradient of radius CP = $-\frac{12}{5}$.

Tangent gradient = $\frac{5}{12}$: $-5x + 12y = 118$.

$x = 0$: $y = \frac{59}{6}$. $y = 0$: $x = -\frac{118}{5}$.

Area = ½ × $\frac{118}{5}$ × $\frac{59}{6}$.

Answer: $\frac{3481}{30}$

3 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = e^x + 2$ to find the exact value of

$$\int_0^{\ln 4} \frac{e^x}{e^x + 2}\,dx$$

$du = e^x\,dx$; limits $x = 0$, $\ln 4$ become $u = 3, 6$

Integral = $\int_3^6 \frac{1}{u}\,du = [\ln u]$

Evaluate the limits

Answer: ln 2

4 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (1 kg) lies on a rough table with $\mu = 0.25$, and is joined by a light string over a smooth pulley (see diagram) to particle B (6 kg) on a smooth slope, $\sin\alpha = \frac{4}{5}$.

Released from rest, the string is cut when B has moved 0.5 m. Find the further distance A slides before stopping.

A: $T - 0.25 \times 1 \times 9.8 = 1\,a$. B: $6 \times 9.8 \times \frac{4}{5} - T = 6\,a$

Adding: $9.8(6 \times \frac{4}{5} - 0.25 \times 1) = 7\,a$, so $a = 6.37$

$T = 1(6.37 + 0.25 \times 9.8) = 8.82$

When cut: $v^2 = 2 \times 6.37 \times 0.5 = 6.37$. Then A decelerates at $\mu g = 2.45$

Further distance = $\frac{6.37}{2 \times 2.45} = 1.3$

Answer: 1.3 m

5 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 6 red counters and some white counters, n in total.

Two are taken without replacement. P(both red) = $\frac{1}{3}$. Find n.

P(both red) = $\frac{6}{n} \times \frac{5}{(n-1)} = \frac{30}{n(n-1)}$

So $n(n-1) = 30 \div \frac{1}{3} = 90$

$n^2 - n - 90 = 0$, so $(n-10)(n+9) = 0$

$n > 0$, so $n = 10$

Answer: n = 10

6 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.

Line l_2 is perpendicular to l_1 at B and meets the

x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{9-6}{-9} = -\frac{1}{3}$.

Gradient of $l_2 = 3$. Equation: $y - 9 = 3x$.

$y = 0$: $x = -3$, so C$(-3, 0)$.

AB = $3\sqrt{10}$, BC = $3\sqrt{10}$: area = ½ × AB × BC.

Answer: C$(-3, 0)$; area 45

7 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2y + xy^2 = 6$.

Find the equation of the normal to C at the point $(2, -3)$,

in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At $(2, -3)$: $dy/dx = -\frac{3}{8}$

Normal gradient = $-1 \div (-\frac{3}{8}) = \frac{8}{3}$

$y - -3 = \frac{8}{3}(x - 2)$, then clear fractions

Answer: $8x - 3y - 25 = 0$

8 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class S has 12 students with mean mark 33 and standard deviation 5. Class R has 15 students with mean mark 60 and standard deviation 8.

(a) Find the mean mark of all 27 students.

(b) Find the standard deviation of all 27 students.

Σx for S = 12 × 33 = 396, Σx for R = 15 × 60 = 900

Combined mean = $\frac{1296}{27} = 48.0$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: 12(25 + 1089) + 15(64 + 3600) = 68328

Variance = $\frac{68328}{27} - 48.000^2 = 226.67$, so $\sigma = 15.06$

Answer: (a) 48.0 (b) 15.1

9 Slide down a rough slope then along rough ground (CIE)

A particle starts from rest at the top of a rough slope of length 8 m ($\sin\alpha = \frac{3}{5}$, $\mu = 0.4$), then slides along rough horizontal ground with the same μ.

Use energy to find (a) its speed at the foot of the slope (3 s.f.), (b) the distance it slides on the ground. Take $g = 9.8$ m s^{-2}.

On the slope: $\frac{1}{2} m v^2 = m g \times 8 \sin\alpha - \mu m g \cos\alpha \times 8$

$v^2 = 2 \times 9.8 \times 8(\frac{3}{5} - 0.4 \times \frac{4}{5}) = 43.904$, so $v = 6.626$

On the ground: $\frac{1}{2} m v^2 = \mu m g x$, so $x = \frac{v^2}{2 \mu g} = \frac{43.904}{7.84} = 5.6$

Answer: (a) 6.63 m s^{-1} (b) 5.6 m

10 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_5 2$ is irrational.

Assume $\log_5 2 = \frac{m}{n}$, m, n positive integers.

Then $5^{\frac{m}{n}} = 2$, so $5^m = 2^n$.

Prime factors of 5^m and 2^n must be identical,

but 2 divides 2^n but 2 does not divide 5^m.

This is a contradiction, so $\log_5 2$ is irrational.

Answer: Proved