Day 21 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 5^x - 4x$ has one stationary point.
Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 5^x \ln 5 - 4 = 0$

$5^x = \frac{4}{\ln 5}$, $x = \frac{1}{\ln 5} \ln(\frac{4}{\ln 5}) = 0.566$

$d^2y/dx^2 = 5^x(\ln 5)^2 > 0$ for all x: minimum

$y = \frac{4}{\ln 5} - 4 \times x = 0.223$

Answer: (0.566, 0.223)

2 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{4 - 4\sqrt{5}}{1 - \sqrt{5}}$ in the form $p + q\sqrt{5}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(1 + \sqrt{5})$.
Numerator: $(4 - 4\sqrt{5})(1 + \sqrt{5}) = -16$.
Denominator: $1^2 - 1^2(5) = -4$.
So the fraction = $\frac{-16}{-4}$.

Answer: 4

3 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 15 kg is at rest on rough horizontal ground ($\mu = 0.1$). A horizontal force of 25.2 N acts on it for 7 s and is then removed.
Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take $g = 9.8$ m s^{-2}.

Friction = $\mu mg = 0.1 \times 15 \times 9.8 = 14.7$ N
While pushed: $25.2 - 14.7 = 15a$, so $a = 0.7$
$v = 0.7 \times 7 = 4.9$; $s_1 = \frac{1}{2} \times 0.7 \times 7^2 = 17.15$

After release: deceleration = $\mu g = 0.98$, so $s_2 = \frac{4.9^2}{2 \times 0.98} = 12.25$

Total distance = 17.15 + 12.25

Answer: (a) 4.9 m s^{-1} (b) 29.4 m

4 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(61, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 67)$.

The area 0.5 leaves 0.500 in total in the tails, 0.250 in each

$P(X < 57) = 0.250$ so $\frac{57 - 61}{\sigma} = -0.6745$

$\sigma = \frac{4}{0.6745} = 5.930$

$P(X > 67) = P(Z > 1.012) = 0.156$

Answer: (a) 5.93 (b) 0.156

5 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1500 kg tows a trailer of mass 300 kg along a horizontal road, with driving force 1520 N. Resistances are 800 N on the car and 180 N on the trailer.

(a) Find the tension in the tow bar. At 4.8 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1520 - 800 - 180}{1800} = 0.3$

Trailer: $T - 180 = 300 \times 0.3$, so $T = 270$

After the break the trailer decelerates at $\frac{180}{300} = 0.6$, and stops after $t = \frac{4.8}{0.6} = 8$ s

Trailer travels $\frac{4.8}{2} \times 8 = 19.2$ m

Car: $a = \frac{1520 - 800}{1500} = 0.48$, so $s = 4.8 \times 8 + ½ \times 0.48 \times 8^2 = 53.76$ m

Distance apart = $53.76 - 19.2$

Answer: (a) 270 N (b) 34.56 m

6 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 19 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 53,
(b) the least number of rows for at least 1184 seats.

Arithmetic: $a = 19$, $d = 3$.

(a) $u_{53} = 19 + 52 \times 3 = 175$.

(b) $S_n = \frac{n}{2}(38 + 3n - 3) = \frac{n(3n + 35)}{2} \geq 1184$.

$S_{22} = 1111 < 1184$, $S_{23} = 1196 \geq 1184$.

Answer: (a) 175 (b) 23 rows

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3x^3\sqrt{7 - x}$, for $0 < x < 7$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{3x^2(42 - 7x)}{2\sqrt{7 - x}}$

$\frac{dy}{dx} = 0$: $42 - 7x = 0$, so $x = 6$

$y = 3 \times 6^3 \times 1$

Gradient changes from positive to negative: a maximum

Answer: (6, 648), maximum

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/4} \cos^4 2x \, dx$

Use $\sin^2 A$ or $\cos^2 A$ twice: write as $(\ldots)^2$ with the double angle formula

Integrate term by term

Substitute the limits

Answer: $\frac{3}{32}\pi$

9 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x + 2) the remainder is −11; when divided by (x − 4) the remainder is 1. Find the remainder when f(x) is divided by (x + 2)(x − 4).

f(x) = (x + 2)(x − 4)Q(x) + px + q.
f(−2) = −11: −2p + q = −11.
f(4) = 1: 4p + q = 1.
Subtracting: −6p = −12, so p = 2, q = −7.
Answer: 2x − 7

10 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 7 times in round 1 and 9 times in round 2.
Given that exactly 4 wins occur altogether, find the probability that exactly 1 of them were in round 1.

Let X ~ B(7, p), Y ~ B(9, p) be independent; X + Y ~ B(16, p)

$$P(X = 1 \mid X + Y = 4) = \frac{P(X = 1)P(Y = 3)}{P(X + Y = 4)}$$

$$= \frac{C(7,1)C(9,3)\, p^4(1-p)^{12}}{C(16,4)\, p^4(1-p)^{12}}$$

The powers of p and (1 − p) cancel, leaving $\frac{7 \times 84}{1820}$

Answer: $\frac{21}{65} = 0.323$