Day 209 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 19.2 m, at a horizontal distance of 29 m from O.

Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take g = 9.8 m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$19.2 = 29 - \frac{9.8 \times 29^2}{u^2}$, so $u^2 = \frac{9.8 \times 841}{9.8} = 841$

u = 29

Greatest height = $\frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{841}{39.2}$

Answer: (a) 29 m s^{-1} (b) 21.5 m

2 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\sqrt{2}$ is irrational.

Assume $\sqrt{2} = \frac{a}{b}$ with a, b integers and no common factor.

Then $2b^2 = a^2$, so a^2 is a multiple of 2.
As 2 is prime, a is a multiple of 2: let a = 2k.
Then $2b^2 = 4k^2$, so $b^2 = 2k^2$, and b is a multiple of 2.
a and b have common factor 2: a contradiction. Hence $\sqrt{2}$ is irrational.

Answer: Proved

3 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^2 \frac{2x + 1}{x^2 + x + 5}\,dx$$

Numerator = × derivative of denominator
Integral = $[\ln(x^2 + x + 5)]$
= (ln 11 − ln 5)

Answer: ln 11 − ln 5

4 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 15 pairs of values: $\Sigma x = 45$, $\Sigma y = 345$, $\Sigma x^2 = 271$, $\Sigma xy = 1443$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between −1 and 8. Estimate y when x = 9 and comment on its reliability.

$S_{xx} = 271 - \frac{45^2}{15} = 136$

$S_{xy} = 1443 - \frac{45 \times 345}{15} = 408$

$b = \frac{S_{xy}}{S_{xx}} = \frac{408}{136} = 3$

$a = \bar{y} - b\,\bar{x} = 23 - (3)(3) = 14$
$y = 14 + 3x$
At x = 9: y = 41. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) y = 14 + 3x (b) 41, unreliable (extrapolation)

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5 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2y + xy^2 = -30$.
Find the equation of the tangent to C at the point $(-3, -2)$,
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At $(-3, -2)$: $dy/dx = -\frac{16}{21}$

$y - -2 = (-\frac{16}{21})(x - -3)$, then clear fractions

Answer: 16x + 21y + 90 = 0

6 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each component is faulty with probability 0.2, independently.
Find the least number n of components that must be tested so that the probability of finding at least two faulty is greater than 0.95.

Let $X \sim B(n, 0.2)$. $P(X \geq 2) = 1 - 0.8^n - n(0.2)(0.8)^{n-1}$
This has no simple algebraic solution, so test values of n
n = 21: $P(X \geq 2) = 0.9424$, which is not greater than 0.95
n = 22: $P(X \geq 2) = 0.9520$, which is greater than 0.95
$P(X \geq 2)$ increases with n, so the least value is 22
Answer: n = 22

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2\sqrt{20 - x}$, for $0 < x < 20$, has one
stationary point. Find its exact coordinates and
determine its nature.

$dy/dx = \frac{2x(80 - 5x)}{2\sqrt{20 - x}}$
$dy/dx = 0$: $80 - 5x = 0$, so $x = 16$
$y = 2 \times 16^2 \times 2$
Gradient changes from positive to negative: a maximum
Answer: (16, 1024), maximum

8 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 - 6x + 6y - 7 = 0$
passes through P(7, 0). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre $(3, -3)$. Gradient of radius CP = $\frac{3}{4}$.

Tangent gradient = $-\frac{4}{3}$: $4x + 3y = 28$.

$x = 0$: $y = \frac{28}{3}$. $y = 0$: $x = 7$.

Area = ½ × 7 × $\frac{28}{3}$.

Answer: $\frac{98}{3}$

9 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 150 N is pulled at constant speed along rough horizontal ground ($\mu = \frac{7}{24}$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{4}{3}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 150$. Horizontal: $P\cos\theta = \mu R = \frac{7}{24}(150 - P\sin\theta)$

$P(\cos\theta + \frac{7}{24}\sin\theta) = \frac{7}{24} \times 150$, so $P = 43.75 \div (\cos\theta + \frac{7}{24}\sin\theta)$

(a) $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$: $P = 43.75 \div \frac{5}{6} = 52.5$

(b) P is least when $\cos\theta + \frac{7}{24}\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + \frac{7}{24}^2)} = \frac{25}{24}$,

which occurs when $\tan\theta = \frac{7}{24}$. Then $P = 43.75 \div \frac{25}{24} = 42$ and $\theta = 16.3°$

Answer: (a) 52.5 N (b) 42 N at θ = 16.3°

10 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 5x - 12$.

Show that $(x - 3)$ is a factor of f(x), and that the equation $f(x) = 0$ has exactly one real root.

$f(3) = 27 - 15 - 12 = 0$, so $(x - 3)$ is a factor.

$f(x) = (x - 3)(x^2 + 3x + 4)$.

For $x^2 + 3x + 4 = 0$: $b^2 - 4ac = 9 - 16 = -7 < 0$.

No real roots from the quadratic, so the only real root is $x = 3$.

Answer: Proved; the only real root is x = 3