Day 208 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (2 m, 100 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{4}{3}$, where α is the angle between string and rod. A 40 N particle hangs at P, AP = 0.5 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Moments about A: $T \sin\alpha \times 2 = 100 \times 1 + 40 \times 0.5$

$T \times \frac{4}{5} \times 2 = 120$, so $T = 75$

Horizontal: $H = T\cos\alpha = 45$. Vertical: $V + T\sin\alpha = 100 + 40$, so $V = 80$

Force $= \sqrt{(45^2 + 80^2)} = 91.8$, angle $= \tan^{-1}(\frac{80}{45}) = 60.6°$ above the horizontal

Answer: (a) 75 N (b) 91.8 N at 60.6° above the horizontal (away from the wall)

2 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 6 \times 2^{x+1} + 20 = 0$,

giving each answer exactly and, where not an integer, to 3 s.f.

$2^{x+1} = 2 \times 2^x$. Let $u = 2^x$: $u^2 - 12u + 20 = 0$

$(u - 10)(u - 2) = 0$, so $2^x = 10$ or $2^x = 2$

Take logarithms: $x = \log_2 2 = 1$; $x = \log_2 10 = 3.32$

Answer: $x = \log_2 2 = 1$; $x = \log_2 10 = 3.32$

3 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.

Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle $= \frac{1}{2} \times 4 \times k = 1$, so $k = \frac{1}{2}$

(b) The mean of a triangular distribution is $\frac{0 + 3 + 4}{3} = \frac{7}{3}$

$F(3) = \frac{3}{4} = \frac{3}{4} \geq \frac{1}{2}$, so the median is in the left part

$F(x) = \frac{x^2}{12} = \frac{1}{2}$, so $x = \sqrt{6} = 2.449$

Answer: (a) $k = \frac{1}{2}$ (b) $E(X) = \frac{7}{3}$ (c) 2.45

4 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 4 strips to estimate

$\int_1^3 \sqrt{x}e^x \, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{2}$; ordinates at $x = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, 3$

Sum $= \frac{h}{2}[y_0 + y_4 + 2(y_1 + \ldots + y_3)] = 26.98$

$f''(x) > 0$ on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 26.98; overestimate

5 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n - 3$, $u_1 = -4$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 1144$.

$u_n = u_1 + \sum_{k=1}^{n-1}(k - 3) = -4 + 1 \times \frac{(n-1)n}{2} - 3(n-1)$.

$= \frac{1}{2}n^2 - \frac{7}{2}n - 1$.

So $a = \frac{1}{2}$, $b = -\frac{7}{2}$, $c = -1$.

$u_{51} = 1121 \leq 1144$ and $u_{52} = 1169 > 1144$.

Answer: (a) $a = \frac{1}{2}$, $b = -\frac{7}{2}$, $c = -1$ (b) $n = 52$

6 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(-3\mathbf{i} - \mathbf{j})$ m with velocity $(-12\mathbf{i} - 7\mathbf{j})$ m s^{-1}. At t = 8, P has velocity $(20\mathbf{i} + 17\mathbf{j})$ m s^{-1}.

Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{j})$, (c) the position vector of P at that time.

$a = \frac{v - u}{t} = \frac{(20\mathbf{i} + 17\mathbf{j}) - (-12\mathbf{i} - 7\mathbf{j})}{8} = (4\mathbf{i} + 3\mathbf{j})$

$v = (-12\mathbf{i} - 7\mathbf{j}) + t(4\mathbf{i} + 3\mathbf{j})$

Parallel to $(\mathbf{j})$: $(4t - 12) \times 1 = (3t - 7) \times 0$, giving $t = 3$

$r = (-3\mathbf{i} - \mathbf{j}) + 3(-12\mathbf{i} - 7\mathbf{j}) + \frac{1}{2} \times 3^2 (4\mathbf{i} + 3\mathbf{j}) = (-21\mathbf{i} - 8.5\mathbf{j})$

Answer: (a) $(4\mathbf{i} + 3\mathbf{j})$ m s^{-2} (b) $t = 3$ (c) $(-21\mathbf{i} - 8.5\mathbf{j})$ m

7 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations

$x = -1 + 5\cos t$, $y = 4 + 5\sin t$.

Find its Cartesian equation.

$\cos t = \frac{x + 1}{5}$, $\sin t = \frac{y - 4}{5}$.

Use $\cos^2 t + \sin^2 t = 1$:

$(x + 1)^2 + (y - 4)^2 = 25$.

Answer: $(x + 1)^2 + (y - 4)^2 = 25$

8 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle with equation $(x + 3)^2 + (y - 1)^2 = 100$

passes through P(5, 7). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre $(-3, 1)$. Gradient of radius CP = $\frac{3}{4}$.

Tangent gradient = $-\frac{4}{3}$: $8x + 6y = 82$.

$x = 0$: $y = \frac{41}{3}$. $y = 0$: $x = \frac{41}{4}$.

Area = $\frac{1}{2} \times \frac{41}{4} \times \frac{41}{3}$.

Answer: $\frac{1681}{24}$

9 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 6 times in round 1 and 8 times in round 2.

Given that exactly 5 wins occur altogether, find the probability that exactly 2 of them were in round 1.

Let $X \sim B(6, p)$, $Y \sim B(8, p)$ be independent; $X + Y \sim B(14, p)$

$P(X = 2 \mid X + Y = 5) = \dfrac{P(X = 2)P(Y = 3)}{P(X + Y = 5)}$

$= \dfrac{C(6,2)C(8,3)\, p^5(1-p)^9}{C(14,5)\, p^5(1-p)^9}$

The powers of p and $(1 - p)$ cancel, leaving $\dfrac{15 \times 56}{2002}$

Answer: $\dfrac{60}{143} = 0.420$

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ 8 \end{pmatrix} + s\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -3 \\ 7 \\ 9 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $1 = -3 + 2t$; $-6 + 3s = 7 - 2t$; $8 - s = 9 - 2t$

Solving the first two: $s = 3$, $t = 2$; the third equation is satisfied

Intersection: (1, 3, 5)

$\cos\varphi = \dfrac{|-4|}{\sqrt{10}\sqrt{12}} = 0.3651$, $\varphi = 68.6°$

Answer: Lines meet at (1, 3, 5); angle 68.6°