Day 207 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Rough slope: up then back down (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 10 m s^{-1} up a line of greatest slope of a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{4}{5}$ and $\mu = 0.65$.

Find (a) the distance it travels up the plane, to 3 s.f., (b) its speed on returning to its starting point. Take g = 9.8 m s^{-2}.

$\cos\alpha = \frac{3}{5}$. Up the plane: R = mg cos α, so $a_{up} = g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{4}{5} + 0.65 \times \frac{3}{5}) = 11.662$

$0 = 10^2 - 2 \times 11.662 \times d$, so $d = \frac{100}{23.324} = 4.287$

Down the plane: $a_{down} = g(\sin\alpha - \mu\cos\alpha) = 9.8(\frac{4}{5} - 0.65 \times \frac{3}{5}) = 4.018$

$v^2 = 2 \times 4.018 \times 4.2874 = 34.45$, so v = 5.87

Answer: (a) 4.29 m (b) 5.87 m s^{-1}

2 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^{\pi/2} \frac{\cos x}{4 + \sin x}\,dx$$

Numerator is the derivative of the denominator
Integral = [ln(4 + sin x)]
= ln 5 − ln 4

Answer: $-2\ln 2 + \ln 5$

3 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 3 and variance 2.25.
(a) Find n and p.
(b) Find P(X = 5), correct to 3 significant figures.

np = 3 and np(1 − p) = 2.25

Divide: $1 - p = 2.25 \div 3 = \frac{3}{4}$, so $p = \frac{1}{4}$

$n = 3 \div \frac{1}{4} = 12$

$P(X = 5) = C(12, 5)\,(\frac{1}{4})^5\,(\frac{3}{4})^7 = 0.103$

Answer: (a) n = 12, p = $\frac{1}{4}$ (b) 0.103

4 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (1, 4, −2), (−2, 0, −1), (3, 0, 1).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, $BC = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$; BA · BC = 13

$|BA| = \sqrt{26}$, $|BC| = \sqrt{29}$

$\cos ABC = \frac{13}{\sqrt{26}\sqrt{29}} = 0.4734$, so ABC = 61.7°

$|BA \times BC| = \left|\begin{pmatrix} 8 \\ -11 \\ -20 \end{pmatrix}\right| = \sqrt{585}$

Area = $\frac{1}{2}|BA \times BC|$

Answer: ABC = 61.7°, area = $\frac{3}{2}\sqrt{65}$

5 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 28 and upper quartile 37. An outlier is a value more than 2 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 12 observations have mean 29. The largest is 58, an outlier, and it is removed. Find the new mean.

IQR = 37 − 28 = 9, so 2 × IQR = 18

Lower limit = 28 − 18 = 10, upper limit = 37 + 18 = 55

$\Sigma x = 12 \times 29 = 348$, so without 58: $\Sigma x = 290$

New mean = $\frac{290}{11}$

Answer: (a) 10 to 55 (b) 26.36

6 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 8 seats. Each row has 4 more seats than the row in front. Find

(a) the number of seats in row 41,

(b) the least number of rows for at least 1229 seats.

Arithmetic: a = 8, d = 4.

(a) $u_{41} = 8 + 40 \times 4 = 168$.

(b) $S_n = \frac{n}{2}(16 + 4n - 4) = \frac{n(4n + 12)}{2} \geq 1229$.

$S_{23} = 1196 < 1229$, $S_{24} = 1296 \geq 1229$.

Answer: (a) 168 (b) 24 rows

7 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 3$ to find the exact value of

$\int_0^1 x(x^2 + 3)^4 \, dx$

du = 2x dx; limits x = 0, 1 become u = 3, 4

Integral = $\frac{1}{2}\int_3^4 u^4 \, du = [\frac{u^5}{10}]$

Evaluate the limits

Answer: $\frac{781}{10}$

8 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{6x^2 - 37x + 54}{(x - 2)(x - 4)^2}$ in the form

$\frac{A}{(x - 2)} + \frac{B}{(x - 4)} + \frac{C}{(x - 4)^2}$.

$6x^2 - 37x + 54 \equiv A(x - 4)^2 + B(x - 2)(x - 4) + C(x - 2)$.

Put x = 2: $4 = A(-2)^2$, so A = 1.

Put x = 4: 2 = C(2), so C = 1.

Compare x^2: 6 = 1A + 1B, so B = 5.

Answer: A = 1, B = 5, C = 1

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 4 kg lies on a smooth horizontal table, with a block of mass 4 kg on it ($\mu = 0.4$ between them). A horizontal force P N acts on the block.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 78.4. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The plank is accelerated only by friction: $F = 4a$

Limiting friction: $F = \mu mg = 0.4 \times 4 \times 9.8 = 15.68$, so $a = \frac{15.68}{4} = 3.92$

P_max = 8 × 3.92 = 31.36

P = 78.4 > 31.36, so the block slips. Plank: $a = \frac{15.68}{4} = 3.92$

Block: $78.4 - 15.68 = 4a$, so $a = 15.68$

Answer: (a) 31.36 N (b) block 15.68 m s^{-2}, plank 3.92 m s^{-2}

10 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{3x - 18}{x - 3}$.

(a) Write down the equations of the asymptotes of C.

(b) The line $y = x + k$ is a tangent to C.

Find the possible values of k.

(a) Vertical: $x = 3$. As $x \to \infty$, $y \to 3$, so $y = 3$.

(b) $3x - 18 = (x + k)(x - 3)$.

$x^2 + (k - 6)x + (-3k + 18) = 0$.

Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 - 36 = 0$.

Answer: (a) x = 3, y = 3 (b) k = −6 or k = 6