Day 206 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 250 kg, carrying a passenger of mass 90 kg, is supported by a vertical cable. The lift is moving upwards and decelerating at 1.6 m s^{-2}.

Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is downwards

Passenger: $90 \times 9.8 - R = 90 \times 1.6$

$R = 90 \times (9.8 - 1.6) = 738$

Lift and passenger: $340 \times 9.8 - T = (250 + 90) \times 1.6$

$T = 340 \times 8.2 = 2788$

Answer: (a) 738 N (b) 2788 N

2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.

Initially P = 200 and after 2 hours P = 300.

Find k, and the time for the population to reach 2000, to 3 s.f.

$P_0 = 200$. $200\,e^{2k} = 300$ so $e^{2k} = \frac{3}{2}$

$k = \frac{1}{2} \ln \frac{3}{2} = 0.203$

$200\,e^{kt} = 2000$ so $e^{kt} = 10$, $t = \frac{1}{k} \ln 10$

t = 11.4 hours

Answer: k = 0.203, t = 11.4 hours

3 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A seed supplier says that 40% of its seeds germinate. 30 seeds are planted, and 10 germinate.

Test, at the 1% significance level, the claim that the proportion is lower.

Give your conclusion in context.

H_0: p = 0.4, H_1: p < 0.4. Let X ~ B(30, 0.4)

$P(X \leq 10) = 0.291$

0.291 is greater than 1%, so do not reject H_0

There is insufficient evidence that the proportion is lower than the claimed 0.4

Answer: Do not reject H_0: p-value 0.291 > 1%

4 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 4 strips to estimate

$\int_0^1 e^{x^2}\,dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{4}$; ordinates at x = 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, …, 1

Sum = $\frac{h}{2}[y_0 + y_4 + 2(y_1 + \ldots + y_3)] = 1.491$

f''(x) > 0 on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 1.491; overestimate

5 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A pile of logs has 9 logs in its bottom row. Each row above has 1 fewer logs than the row below, and the pile has as many rows as possible, each with at least one log. Find the number of rows and the total number of logs.

Row n has $9 - 1(n - 1) = -n + 10$ logs; need this ≥ 1.

$1(n - 1) \leq 8$, so $n \leq 9$: 9 rows.

Top row = 1 logs. $S = \frac{9}{2}(9 + 1) = 45$.

Answer: 9 rows, 45 logs

6 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(57, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 77)$.

The area 0.95 leaves 0.050 in total in the tails, 0.025 in each

$P(X < 47) = 0.025$ so $\frac{47 - 57}{\sigma} = -1.9600$

$\sigma = \frac{10}{1.9600} = 5.102$

$P(X > 77) = P(Z > 3.920) = 0.0000443$

Answer: (a) 5.10 (b) 0.0000443

7 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 3x^3 + ax^2 + bx + 48$, where a and b are constants.
$(x + 2)$ is a factor of f(x), and f(x) leaves remainder 0 when divided by $(x - 4)$.
Find a and b, and hence solve $f(x) = 0$.

$f(-2) = 0$: $-24 + 4a - 2b + 48 = 0$, so $4a - 2b + 24 = 0$.
$f(4) = 0$: $16a + 4b + 240 = 0$.
Solving: $a = -12$, $b = -12$.
$f(x) = 3x^3 - 12x^2 - 12x + 48 = 3(x + 2)(x - 2)(x - 4)$
Answer: a = −12, b = −12; x = −2, 2, 4

8 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 135 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{7}{24}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{4}{3}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 135\cos\alpha$, so $R = 135 \times \frac{24}{25} - P\sin\beta$

Along the plane: $P\cos\beta = 135\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 135(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{3}{5} + \frac{3}{4} \times \frac{4}{5}) = 135(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25})$, so $P = 112.5$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_{min} = 135(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25}) \div \frac{5}{4}$

Answer: (a) 112.5 N (b) 108 N

9 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(4 + 6x)^{-\frac{3}{2}}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(4 + 6x)^{-\frac{3}{2}} = 4^{-\frac{3}{2}}(1 + \frac{3}{2}x)^{-\frac{3}{2}} = \frac{1}{8}(1 + u)^{-\frac{3}{2}}$, $u = \frac{3}{2}x$.

$(1 + u)^{-\frac{3}{2}} = 1 + (-\frac{3}{2})u + \frac{15}{8}u^2 + (-\frac{35}{16})u^3 + ...$

Substitute u and multiply by $\frac{1}{8}$.

Valid when $|u| < 1$, i.e. $|x| < \frac{2}{3}$.

Answer: $\frac{1}{8} - \frac{9}{32}x + \frac{135}{256}x^2 - \frac{945}{1024}x^3 + ...$; $|x| < \frac{2}{3}$

10 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the points of inflection of

$y = \frac{1}{x^2 + 12}$.

$dy/dx = -\frac{2x}{(x^2 + 12)^2}$

$d^2y/dx^2 = \frac{2(3x^2 - 12)}{(x^2 + 12)^3}$

Zero when $x^2 = 4$, so $x = \pm 2$, and the sign changes either side

$y = \frac{1}{16} = \frac{1}{16}$

Answer: $(2, \frac{1}{16})$ and $(-2, \frac{1}{16})$