Day 204 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 4x^2 + 10x + 12$.
Show that $(x + 2)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-2) = -8 + 16 - 20 + 12 = 0$, so $(x + 2)$ is a factor.

$f(x) = (x + 2)(x^2 + 2x + 6)$.

For $x^2 + 2x + 6 = 0$: $b^2 - 4ac = 4 - 24 = -20 < 0$.

No real roots from the quadratic, so the only real root is $x = -2$.

Answer: Proved; the only real root is $x = -2$

2 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $3\cos\theta + 4\sin\theta = 3$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3}$, so $\alpha = 53.1°$

So $3\cos\theta + 4\sin\theta \equiv 5\cos(\theta - 53.1°)$

$\cos(\theta - \alpha) = \frac{3}{5}$

$\theta - \alpha = \pm 53.1°$ (using $\alpha = 53.1°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) $R = 5$, $\alpha = 53.1°$ (b) $\theta = 106.3°, 360.0°$

3 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - 2y = 3$
$2x^2 - xy + y^2 = 18$

From the first equation $x = 3 + 2y$.

Substitute into the second equation and collect terms:

this gives $7y^2 + 21y = 0$.

$y = -3$ or $y = 0$, giving $x = -3$ or $x = 3$.

Answer: $(x, y) = (-3, -3)$ and $(3, 0)$

4 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 104 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{5}{12}$ and $\tan\beta = \frac{12}{5}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 156 N.

Horizontally: $T_1\cos\alpha = T_2\cos\beta$, so $T_1 \times \frac{12}{13} = T_2 \times \frac{5}{13}$

Vertically: $T_1\sin\alpha + T_2\sin\beta = 104$, and $T_2 = \frac{12}{5}T_1$

$T_1(\frac{5}{13} + \frac{12}{5} \times \frac{12}{13}) = 104$, so $T_1 = 40$ and $T_2 = 96$

The tension in BP is the larger: it is 96 when $W = 104$

Tensions are proportional to W, so $W_{max} = 104 \times \frac{156}{96} = 169$

Answer: (a) AP: 40 N, BP: 96 N (b) 169 N

5 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|x - 5| \leq |-4x - 5|$.

Square both sides: $(x - 5)^2 \leq (-4x - 5)^2$.
$(5x)(-3x - 10) \leq 0$.

Critical values $x = 0$ and $x = -\frac{10}{3}$.

The quadratic has negative coefficient of x^2, so choose the outer region.

Answer: $x \leq -\frac{10}{3}$ or $x \geq 0$

6 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(92, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 109)$.

The area 0.95 leaves 0.050 in total in the tails, 0.025 in each

$P(X < 80) = 0.025$ so $\frac{80 - 92}{\sigma} = -1.9600$

$\sigma = \frac{12}{1.9600} = 6.123$

$P(X > 109) = P(Z > 2.777) = 0.00275$

Answer: (a) 6.12 (b) 0.00275

7 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$.
Find the exact area under one arch, for $0 \le \theta \le 2\pi$,
between the curve and the x-axis.

$dx/d\theta = 2(1 - \cos\theta)$

Area $= \int_0^{2\pi} 2^2(1 - \cos\theta)^2 \, d\theta = 4\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, so the integrand integrates to $2\pi + \frac{1}{2} \times 2\pi = 3\pi$

Answer: 12π

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{3}{20 + 5\sin\theta + 12\cos\theta}$

(a) Write $5\sin\theta + 12\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{5^2 + 12^2} = 13$, $\tan\alpha = \frac{12}{5}$, $\alpha = 67.4°$

Denominator $= 20 + 13\sin(\theta + 67.4°)$, which lies between 7 and 33

Greatest $f(\theta) = \frac{3}{7} = \frac{3}{7}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{3}{33} = \frac{1}{11}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 67.4° = 270°$, so $\theta = 202.6°$

Answer: (a) $13\sin(\theta + 67.4°)$ (b) greatest $\frac{3}{7}$, least $\frac{1}{11}$, $\theta = 202.6°$

9 Car on a hill: power and maximum speed (CIE)

A car of mass 1500 kg, with engine power 14.82 kW, climbs a straight road with $\sin\alpha = \frac{1}{20}$ against a constant resistance of 500 N.
Find (a) the greatest steady speed up the hill, (b) the acceleration at 3 m s^{-1}. Take $g = 9.8$ m s^{-2}.

Steady speed: driving force $= 500 + 1500 \times 9.8 \times \frac{1}{20} = 1235$ N

$P = F v$, so $v = \frac{14820}{1235} = 12$

At 3 m s^{-1}: $F = \frac{14820}{3} = 4940$ N

$F - 1235 = 1500\,a$, so $a = \frac{3705}{1500}$

Answer: (a) 12 m s^{-1} (b) 2.47 m s^{-2}

10 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased spinner lands on 'win' with the same unknown probability p each time. It is used 5 times in round 1 and 8 times in round 2.

Given that exactly 6 wins occur altogether, find the probability that exactly 4 of them were in round 1.

Let $X \sim B(5, p)$, $Y \sim B(8, p)$ be independent; $X + Y \sim B(13, p)$

$$P(X = 4 \mid X + Y = 6) = \frac{P(X = 4)P(Y = 2)}{P(X + Y = 6)}$$

$$= \frac{C(5,4)C(8,2)\, p^6(1 - p)^7}{C(13,6)\, p^6(1 - p)^7}$$

The powers of p and $(1 - p)$ cancel, leaving $\frac{5 \times 28}{1716}$

Answer: $\frac{35}{429} = 0.0816$