Day 203 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 16 cm and its area is 15 cm^2.
Find the two possible pairs of values of r and θ.

Arc = rθ, so $2r + r\theta = 16$ and $\frac{1}{2}r^2\theta = 15$

$r\theta = 16 - 2r$, so $\frac{1}{2}r(16 - 2r) = 15$

$r^2 - 8r + 15 = 0$, so $(r - 3)(r - 5) = 0$

r = 3: rθ = 10, $\theta = \frac{10}{3}$ r = 5: rθ = 6, $\theta = \frac{6}{5}$

Answer: r = 3, $\theta = \frac{10}{3}$ or r = 5, $\theta = \frac{6}{5}$

2 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 6 m s^{-1} from the top of a cliff. The top of the cliff is 26.1 m above the sea. Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-26.1 = 6t - 4.9t^2$, so $4.9t^2 - 6t - 26.1 = 0$

$t = \frac{6 + \sqrt{(6^2 + 19.6 \times 26.1)}}{9.8} = \frac{6 + 23.4}{9.8} = 3$

Height risen above the cliff top = $\frac{6^2}{(2 \times 9.8)} = 1.84$ m

Total distance = 2 × 1.84 + 26.1

Answer: (a) 3 s (b) 29.8 m

3 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 4 and variance 3.2.
(a) Find n and p.
(b) Find P(X = 6), correct to 3 significant figures.

$np = 4$ and $np(1 - p) = 3.2$

Divide: $1 - p = 3.2 \div 4 = \frac{4}{5}$, so $p = \frac{1}{5}$

$n = 4 \div \frac{1}{5} = 20$

$P(X = 6) = C(20, 6) (\frac{1}{5})^6 (\frac{4}{5})^{14} = 0.109$

Answer: (a) n = 20, $p = \frac{1}{5}$ (b) 0.109

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2\text{cosec}^2\theta - \cot\theta - 8 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $2\cot^2\theta - \cot\theta - 6 = 0$
$(2\cot\theta + 3)(\cot\theta - 2) = 0$

$\cot\theta = -\frac{3}{2}$ or $\cot\theta = 2$

$\tan\theta = -\frac{2}{3}$ or $\frac{1}{2}$, so θ = 146.3° and 26.6° (+180°)

Answer: θ = 26.6°, 146.3°, 206.6°, 326.3°

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5 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n(n + 8)(n + 16)(n + 24) + 4096$ is a perfect square for every integer n.

Pair the factors: $n(n + 24) = n^2 + 24n$ and $(n + 8)(n + 16) = n^2 + 24n + 128$.
Let $u = n^2 + 24n$. Expression $= u(u + 128) + 4096$
$= u^2 + 128u + 4096 = (u + 64)^2$
$= (n^2 + 24n + 64)^2$, a perfect square.
Answer: Proved

6 Poisson hypothesis test (CIE)

Faults in a cable occur at random at a mean rate of 4 per day. After a modification, 0 faults are found in 1 day. Test at the 5% level whether the rate of faults has decreased.

H_0: $\lambda = 4$ per day, H_1: $\lambda < 4$ per day
Over 1 day: $X \sim Po(4)$
$P(X \leq 0) = 0.0183$
$0.0183 < 5\%$, so reject H_0
There is sufficient evidence that the rate has decreased
Answer: Reject H_0 (p-value 0.0183)

7 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(9 + x)^{\frac{1}{2}}$ up to and including the term in x^2.
(b) Use $x = 0.05$ to estimate $\sqrt{9.05}$ to 5 decimal places.

(a) $(9 + x)^{\frac{1}{2}} = 3(1 + \frac{1}{9}x)^{\frac{1}{2}}$.

$= 3 + \frac{1}{6}x - \frac{1}{216}x^2 + \ldots$
(b) $9 + x$ with $x = 0.05$ gives 9.05.
Substitute: $3 + \frac{1}{6}(0.05) + (-\frac{1}{216})(0.05)^2$.
$= 3.00832$ (5 d.p.).
Answer: (a) $3 + \frac{1}{6}x - \frac{1}{216}x^2$ (b) 3.00832

8 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{1}{(x - 3)} > \frac{3}{(x - 1)}$.

Multiply by the positive quantity $(x - 3)^2(x - 1)^2$:
$1(x - 3)(x - 1)^2 > 3(x - 1)(x - 3)^2$.
Move to one side and factorise: $(x - 3)(x - 1)(-2x + 8) > 0$.
Critical values: 3, 1, 4. Test the four regions.
Answer: $x < 1$ or $3 < x < 4$

9 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 12 \\ 9 \\ -5 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$.

The point P has coordinates (1, 3, −2).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} -11 \\ -6 \\ 3 \end{pmatrix} - t\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$

FP · d = 0: 51 − t(17) = 0, so t = 3

F = (3, 3, 1)

$\mathrm{PF} = \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{13}$

Answer: F = (3, 3, 1), PF = $\sqrt{13}$

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector (14**i** + 26**j**) km relative to a port O, and moves with constant velocity (**i** + 2**j**) km h^{-1}. Ship B leaves O at t = 0 with constant speed 17 km h^{-1} and intercepts A.

Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: (14**i** + 26**j**) + t(**i** + 2**j**). B at time t: t v_B (unknown velocity)

At interception B has travelled 17 t, so |(14**i** + 26**j**) + t(**i** + 2**j**)| = 17 t

Squaring and rearranging: $-284t^2 + 132t + 872 = 0$, which gives t = 2 or a negative value

t = 2 (the other root is negative since B is faster than A)

A is then at (14**i** + 26**j**) + 2(**i** + 2**j**) = (16**i** + 30**j**), so $\mathbf{v}_B = \frac{1}{2}(16\mathbf{i} + 30\mathbf{j}) = (8\mathbf{i} + 15\mathbf{j})$

Answer: (a) 2 h (b) (8i + 15j) km h^{-1}