Day 201 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ab^x$. The graph of lg y against x is the straight line shown.
Find a and b to 3 significant figures.

lg y = lg a + x lg b. The intercept is lg a = 1.2, so $a = 10^{1.2} = 15.85$

Gradient = $\frac{4 - 1.2}{7} = 0.4$

lg b = 0.4 so $b = 10^{0.4} = 2.512$

Answer: a = 15.8, b = 2.51

2 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ladder of length 25 m leans against a vertical wall. Its foot slides away from the wall at $\frac{1}{2}$ m s^{-1}. When the foot is 24 m from the wall, find the rate at which the top is moving down.

$x^2 + y^2 = 625$, so $2x\,dx/dt + 2y\,dy/dt = 0$
When x = 24, y = 7

$dy/dt = -\frac{x}{y}\,dx/dt = -\frac{24}{7} \times \frac{1}{2}$

Answer: $\frac{12}{7}$ m s^{-1}

3 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{5x - 4} = x - 2$.

Square both sides: $5x - 4 = (x - 2)^2$.
$x^2 - 9x + 8 = 0$, so x = 1 or x = 8.
Check in $\sqrt{5x - 4} = x - 2$: x = 8 gives $6 \geq 0$ (valid).
x = 1 gives right-hand side $-1 < 0$, so it is rejected.
Answer: x = 8

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $12\text{cosec}^2\theta - 5\cot\theta - 14 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $12\cot^2\theta - 5\cot\theta - 2 = 0$
$(3\cot\theta - 2)(4\cot\theta + 1) = 0$

$\cot\theta = \frac{2}{3}$ or $\cot\theta = -\frac{1}{4}$

$\tan\theta = \frac{3}{2}$ or -4, so $\theta = 56.3°$ and $104.0°$ $(+180°)$

Answer: θ = 56.3°, 104.0°, 236.3°, 284.0°

5 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 6 m. After each bounce it rises to $\frac{2}{3}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.1 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 6 + \frac{2 \times 6 \times \frac{2}{3}}{1 - \frac{2}{3}} = 30.$

(b) Rebound height after n bounces $= 6(\frac{2}{3})^n < 0.1$.

$n > \frac{\log(0.0166667)}{\log(0.666667)} = 10.10$, so n = 11.

Answer: (a) 30 m (b) 11 bounces

6 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(39, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 49)$.

The area 0.9 leaves 0.100 in total in the tails, 0.050 in each

$P(X < 35) = 0.050$ so $\frac{35 - 39}{\sigma} = -1.6449$

$\sigma = \frac{4}{1.6449} = 2.432$

$P(X > 49) = P(Z > 4.112) = 0.0000196$

Answer: (a) 2.43 (b) 0.0000196

7 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.
Initially P = 50 and after 3 hours P = 75.
Find k, and the time for the population to reach 300, to 3 s.f.

$P_0 = 50$. $50\,e^{3k} = 75$ so $e^{3k} = \frac{3}{2}$

$k = \frac{1}{3} \ln \frac{3}{2} = 0.135$

$50\,e^{kt} = 300$ so $e^{kt} = 6$, $t = \frac{1}{k} \ln 6$

t = 13.3 hours

Answer: k = 0.135, t = 13.3 hours

8 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 35 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{3}{4}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{12}{5}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 35\cos\alpha$, so $R = 35 \times \frac{4}{5} - P\sin\beta$

Along the plane: $P\cos\beta = 35\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 35(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13}) = 35(\frac{3}{5} + \frac{3}{4} \times \frac{4}{5})$, so P = 39

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_{min} = 35(\frac{3}{5} + \frac{3}{4} \times \frac{4}{5}) \div \frac{5}{4}$

Answer: (a) 39 N (b) 33.6 N

9 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A pile of logs has 16 logs in its bottom row. Each row above has 1 fewer logs than the row below, and the pile has as many rows as possible, each with at least one log. Find the number of rows and the total number of logs.

Row n has $16 - 1(n - 1) = -n + 17$ logs; need this ≥ 1.
$1(n - 1) \leq 15$, so $n \leq 16$: 16 rows.

Top row = 1 logs. $S = \frac{16}{2}(16 + 1) = 136$.

Answer: 16 rows, 136 logs

10 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (2.5t^2 - 3t)\mathbf{i} + (-2t^2 + 6t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(2\mathbf{i} - \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (5t - 3)\mathbf{i} + (-4t + 6)\mathbf{j}$

Parallel to $(2\mathbf{i} - \mathbf{j})$: $(5t - 3) \times -1 = (-4t + 6) \times 2$

$t(3) = 9$, so $t = 3$

At $t = 3$: $\mathbf{v} = (12\mathbf{i} - 6\mathbf{j})$, speed $= \sqrt{(12^2 + -6^2)}$

Answer: (a) t = 3 (b) 13.4 m s^{-1}