Day 2 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A van costs £24000 and loses 15% of its value each year.
Find the least whole number of years after which its value is below £3000.

$24000 \times 0.85^n < 3000$, so $0.85^n < \frac{1}{8}$

$n > \frac{\ln \frac{1}{8}}{\ln 0.85} = 12.80$ ($\ln 0.85$ is negative, so the inequality reverses)

$n = 13$

Answer: 13

2 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 20 observations x, $\Sigma(x - 40) = 10$ and $\Sigma(x - 40)^2 = 425$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 40) = \frac{10}{20} = 0.5$, so mean of $x = 40.5$

Variance $= \frac{425}{20} - (0.5)^2 = 21$

$\sigma = \sqrt{21}$

$\Sigma x^2 = \Sigma(x - 40)^2 + 80\,\Sigma(x - 40) + 20 \times 40^2$

$= 425 + 800 + 32000$

Answer: (a) 40.5 (b) 4.58 (c) 33225

3 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (9 kg) and B (9 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\alpha = 30°$, and B is 1.5 m above the ground.
Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 9 \times 9.8 \times \frac{1}{2} = 9a$. B: $9 \times 9.8 - T = 9a$

Adding: $9.8(9 - 9 \times \frac{1}{2}) = 18a$, so $a = 2.45$

$T = 9(9.8 - 2.45) = 66.15$

When B lands: $v^2 = 2 \times 2.45 \times 1.5 = 7.35$

Then A decelerates at $g \sin \alpha = 4.9$: further distance $= \frac{7.35}{2 \times 4.9} = 0.75$

Answer: 0.75 m

4 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 6x^2 + 15x + 14$.
Show that $(x + 2)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-2) = -8 + 24 - 30 + 14 = 0$, so $(x + 2)$ is a factor.

$f(x) = (x + 2)(x^2 + 4x + 7)$.

For $x^2 + 4x + 7 = 0$: $b^2 - 4ac = 16 - 28 = -12 < 0$.

No real roots from the quadratic, so the only real root is $x = -2$.

Answer: Proved; the only real root is $x = -2$

5 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(-\mathbf{i})$ m with velocity $(2\mathbf{i} + 9\mathbf{j})$ m s^{-1}. At t = 8, P has velocity $(-6\mathbf{i} - 15\mathbf{j})$ m s^{-1}.

Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{j})$, (c) the position vector of P at that time.

$a = \frac{v - u}{t} = \frac{(-6\mathbf{i} - 15\mathbf{j}) - (2\mathbf{i} + 9\mathbf{j})}{8} = (-\mathbf{i} - 3\mathbf{j})$

$v = (2\mathbf{i} + 9\mathbf{j}) + t(-\mathbf{i} - 3\mathbf{j})$

Parallel to $(\mathbf{j})$: $(-t + 2) \times 1 = (-3t + 9) \times 0$, giving $t = 2$

$r = (-\mathbf{i}) + 2(2\mathbf{i} + 9\mathbf{j}) + \frac{1}{2} \times 2^2 (-\mathbf{i} - 3\mathbf{j}) = (\mathbf{i} + 12\mathbf{j})$

Answer: (a) $(-\mathbf{i} - 3\mathbf{j})$ m s^{-2} (b) t = 2 (c) $(\mathbf{i} + 12\mathbf{j})$ m

6 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(9, 2) lies on the circle: $(9 - r)^2 + (2 - r)^2 = r^2$.

$r^2 - 22r + 85 = 0$, so r = 5 or r = 17.

The diagram shows P on the smaller circle, so r = 5.

$(x - 5)^2 + (y - 5)^2 = 25$, then expand.

Answer: $x^2 + y^2 - 10x - 10y + 25 = 0$

7 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = \ln y + 2y^2$.

Find dy/dx and d^2y/dx^2 when y = 1.

$dx/dy = 4y + \frac{1}{y} = 5$ when y = 1, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{5}$

$d^2x/dy^2 = 3$ when y = 1

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{3}{5^3}$

Answer: $dy/dx = \frac{1}{5}$, $d^2y/dx^2 = -\frac{3}{125}$

8 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: p = 0.3 against H_1: p > 0.3 uses a sample of 30 at the 5% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact p = 0.4.

X ~ B(30, 0.3). $P(X \geq 14) = 0.0401$ and $P(X \geq 13) = 0.0845$

The first is at most 0.05 and the second is not, so the critical region is $X \geq 14$

Actual significance level = $P(X \geq 14) = 0.0401$

Type II error: H_0 is not rejected, so $X \leq 13$, with X ~ B(30, 0.4)

$P(X \leq 13) = 0.715$

Answer: (a) $X \geq 14$ (b) 0.0401 (c) 0.715

9 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (0, 2, 0), (1, −2, −3), (0, 5, 2).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\text{BA} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, $\text{BC} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix}$; $\text{BA} \cdot \text{BC} = 44$

$|\text{BA}| = \sqrt{26}$, $|\text{BC}| = \sqrt{75}$

$\cos \text{ABC} = \frac{44}{\sqrt{26}\sqrt{75}} = 0.9964$, so $\text{ABC} = 4.9°$

$|\text{BA} \times \text{BC}| = \left|\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}\right| = \sqrt{14}$

$\text{Area} = \frac{1}{2}|\text{BA} \times \text{BC}|$

Answer: ABC = 4.9°, area = $\frac{1}{2}\sqrt{14}$

10 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 2t$, $y = 3t^2$. The line $y = -\frac{9}{2}x - 6$
meets C at A and B. Find the exact length AB.

Substitute: $3t^2 = -9t - 6$.
$3t^2 + 9t + 6 = 0 \Rightarrow t = -2$ or $t = -1$.
A(−4, 12), B(−2, 3).
$\text{AB}^2 = (-2)^2 + 9^2 = 85$.
Answer: AB = $\sqrt{85}$