Day 198 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 20 N is pulled at constant speed along rough horizontal ground ($\mu = 0.75$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{24}{7}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 20$. Horizontal: $P\cos\theta = \mu R = 0.75(20 - P\sin\theta)$

$P(\cos\theta + 0.75\sin\theta) = 0.75 \times 20$, so $P = 15 \div (\cos\theta + 0.75\sin\theta)$

(a) $\cos\theta = \frac{7}{25}$, $\sin\theta = \frac{24}{25}$: $P = 15 \div 1 = 15$

(b) P is least when $\cos\theta + 0.75\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + 0.75^2)} = 1.25$, which occurs when $\tan\theta = 0.75$. Then $P = 15 \div 1.25 = 12$ and $\theta = 36.9°$

Answer: (a) 15 N (b) 12 N at θ = 36.9°

2 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An arithmetic sequence has $u_4 = 15$ and $S_{10} = 225$.

(a) Find the first term and common difference.

(b) Find the least n for which $S_n > 721$.

u_4: $a + 3d = 15$. S_{10}: $10(2a + 9d)/2 = 225$, so $10a + 45d = 225$.

Solving: $a = 0$, $d = 5$.

$S_n = \frac{n}{2}(0 + 5n - 5) = \frac{n(5n - 5)}{2}$.

Test: $S_{17} = 680 \leq 721$, $S_{18} = 765 > 721$.

Answer: (a) a = 0, d = 5 (b) n = 18

3 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class Q has 24 students with mean mark 67 and standard deviation 7. Class R has 10 students with mean mark 31 and standard deviation 5.

(a) Find the mean mark of all 34 students.

(b) Find the standard deviation of all 34 students.

Σx for Q $= 24 \times 67 = 1608$, Σx for R $= 10 \times 31 = 310$

Combined mean $= \frac{1918}{34} = 56.4$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $24(49 + 4489) + 10(25 + 961) = 118772$

Variance $= \frac{118772}{34} - 56.412^2 = 311.01$, so $\sigma = 17.64$

Answer: (a) 56.4 (b) 17.6

4 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(4 - 6x)^{\frac{3}{2}}$ in ascending powers of x up to and including the term in x^3.

State the values of x for which it is valid.

$(4 - 6x)^{\frac{3}{2}} = 4^{\frac{3}{2}}(1 - \frac{3}{2}x)^{\frac{3}{2}} = 8(1 + u)^{\frac{3}{2}}$, $u = -\frac{3}{2}x$.

$(1 + u)^{\frac{3}{2}} = 1 + \frac{3}{2}u + \frac{3}{8}u^2 + (-\frac{1}{16})u^3 + \ldots$

Substitute u and multiply by 8.

Valid when $|u| < 1$, i.e. $|x| < \frac{2}{3}$.

Answer: $8 - 18x + \frac{27}{4}x^2 + \frac{27}{16}x^3 + \ldots$; $|x| < \frac{2}{3}$

5 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = e^x + 2x - 7$.

(a) Show that $f(x) = 0$ has a root α with $1 < \alpha < 2$.

(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places.

$f(1) = -2.28$ and $f(2) = 4.39$: a change of sign, f continuous

$f'(x) = e^x + 2$, so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \dfrac{4.389}{9.389} = 1.5325$

$x_2 = 1.5325 - \dfrac{0.6950}{6.630} = 1.4277$

Answer: $x_1 = 1.5325$, $x_2 = 1.4277$

6 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 2x^2 + 3x + 18$.

Show that $(x + 3)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-3) = -27 + 18 - 9 + 18 = 0$, so $(x + 3)$ is a factor.

$f(x) = (x + 3)(x^2 - x + 6)$.

For $x^2 - x + 6 = 0$: $b^2 - 4ac = 1 - 24 = -23 < 0$.

No real roots from the quadratic, so the only real root is $x = -3$.

Answer: Proved; the only real root is $x = -3$

7 Speed decaying exponentially (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. The velocity of P is v m s^{-1} and its acceleration is $-0.25v$ m s^{-2}. At $t = 0$, $v = 8$. Find (a) the time taken for the speed of P to halve, to 3 s.f., (b) the distance P travels in this time.

$\dfrac{dv}{dt} = -0.25v$: $\int \dfrac{dv}{v} = -0.25\int dt$, so $\ln v = -0.25t + c$, giving $v = 8\,e^{-0.25t}$

$v = 4$: $e^{-0.25t} = \dfrac{1}{2}$, so $t = \dfrac{\ln 2}{0.25}$

$s = \int v\,dt = -\dfrac{8}{0.25}e^{-0.25t} + c$; distance in this time $= \dfrac{8}{0.25}\left(1 - \dfrac{1}{2}\right)$

Answer: (a) 2.77 s (b) 16 m

8 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Write $\dfrac{x + 2}{(x + 3)^2}$ in the form $\dfrac{A}{x + 3} + \dfrac{B}{(x + 3)^2}$.

(b) Find the exact value of $\displaystyle\int_0^1 \dfrac{x + 2}{(x + 3)^2}\,dx$.

$x + 2 \equiv A(x + 3) + B$, so $A = 1$, and $B = -1$

Integral $= [1 \ln(x + 3) - \dfrac{-1}{x + 3}]$

Substitute limits 0 and 1

Answer: (a) $A = 1$, $B = -1$; (b) $-\dfrac{1}{12} + 2\ln 2 - \ln 3$

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ N(μ, 9). H_0: μ = 53 is tested against H_1: μ > 53 using the mean of a sample of size 25 at the 1% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when μ = 54.

Under H_0, sample mean ~ N(53, $\frac{9}{25}$), standard error = 0.6000

z = 2.3263 for the upper tail: critical value = 53 + 2.3263 × 0.6000 = 54.40

Type II error: do not reject H_0, sample mean < 54.40, where mean ~ N(54, 0.6000)

$z = \frac{54.40 - 54}{0.6000} = 0.660$

Probability = 0.745

Answer: (a) sample mean > 54.4 (b) 0.745

10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population N satisfies $\frac{dN}{dt} = -0.01N^2$, with N = 50 when t = 0.

Find N when t = 4, and the time taken for N to halve.

$\int N^{-2}\,dN = -0.01\int dt$ gives $-\frac{1}{N} = -0.01t + c$

t = 0: $c = -\frac{1}{50}$, so $\frac{1}{N} = 0.01t + \frac{1}{50}$

t = 4: $\frac{1}{N} = \frac{1}{25} + \frac{1}{50}$; halving: $0.01t = \frac{1}{50}$

Answer: $N = \frac{50}{3}$; t = 2