Day 196 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{18}{x^2 + 4x + 6}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 + 4x + 6 = (x + 2)^2 + 2$.

$(x + 2)^2 \geq 0$, so the denominator is at least 2, with equality at $x = -2$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{18}{2} = 9$ at $x = -2$.

Answer: greatest value 9 at $x = -2$; range $0 < f(x) \leq 9$

2 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car P passes a point O travelling at a constant speed of 10 m s^{-1}. 5 s later, car Q leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.

Q overtakes P at the point X. Find (a) the time Q takes to reach X, (b) the distance OX, (c) the speed of Q at X.

Let Q travel for t s. P: distance = 10(t + 5); Q: distance = ½ × 3 × t^2

Equate: $3t^2 - 20t - 100 = 0$, so $(t - 10)(3t + 10) = 0$

t = 10 (the other root is negative)

OX = 3 × 10^2 / 2 = 150 m, and speed = 3 × 10 = 30

Answer: (a) 10 s (b) 150 m (c) 30 m s^{-1}

3 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{28x^2 - 126x + 136}{(x - 4)(2x - 4)^2}$ in the form

$\frac{A}{(x - 4)} + \frac{B}{(2x - 4)} + \frac{C}{(2x - 4)^2}$.

$28x^2 - 126x + 136 \equiv A(2x - 4)^2 + B(x - 4)(2x - 4) + C(x - 4)$.

Put $x = 4$: $80 = A(4)^2$, so A = 5.

Put $x = 2$: $-4 = C(-2)$, so C = 2.

Compare x^2: $28 = 4A + 2B$, so B = 4.

Answer: A = 5, B = 4, C = 2

4 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 30 observations x, $\Sigma(x - 20) = 30$ and $\Sigma(x - 20)^2 = 630$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 20) = \frac{30}{30} = 1$, so mean of x = 21

Variance $= \frac{630}{30} - (1)^2 = 20$

$\sigma = \sqrt{20}$

$\Sigma x^2 = \Sigma(x - 20)^2 + 40\,\Sigma(x - 20) + 30 \times 20^2$

$= 630 + 1200 + 12000$

Answer: (a) 21 (b) 4.47 (c) 13830

5 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 13 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 56,
(b) the least number of rows for at least 574 seats.

Arithmetic: $a = 13$, $d = 3$.
(a) $u_{56} = 13 + 55 \times 3 = 178$.
(b) $S_n = \frac{n}{2}(26 + 3n - 3) = \frac{n(3n + 23)}{2} \geq 574$.
$S_{16} = 568 < 574$, $S_{17} = 629 \geq 574$.
Answer: (a) 178 (b) 17 rows

6 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, P(A) = 0.15, P(B) = 0.15 and P(A | B) = 0.6.
(a) Find P(A ∩ B).
(b) Find P(A | B′).

$P(A \cap B) = P(A \mid B) \times P(B) = 0.6 \times 0.15 = 0.09$
$P(A \cup B) = 0.15 + 0.15 - 0.09 = 0.21$
$P(B') = 0.85$
$P(A \cap B') = P(A) - P(A \cap B) = 0.06$
$P(A \mid B') = \frac{0.06}{0.85}$

Answer: (a) 0.09 (b) $\frac{6}{85}$

7 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^1 xe^x \, dx$

Let $u = x$ and $dv/dx = e^x$
Integral = $[uv] - \int v \, du/dx \, dx$
Evaluate between the limits
Answer: 1

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{21 + 8\sin\theta + 15\cos\theta}$
(a) Write $8\sin\theta + 15\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{8^2 + 15^2} = 17$, $\tan\alpha = \frac{15}{8}$, $\alpha = 61.9°$
Denominator $= 21 + 17\sin(\theta + 61.9°)$, which lies between 4 and 38
Greatest $f(\theta) = \frac{4}{4} = 1$, when $\sin(\theta + \alpha) = -1$
Least $f(\theta) = \frac{4}{38} = \frac{2}{19}$, when $\sin(\theta + \alpha) = 1$
$\sin(\theta + \alpha) = -1$: $\theta + 61.9° = 270°$, so $\theta = 208.1°$
Answer: (a) $17\sin(\theta + 61.9°)$ (b) greatest 1, least $\frac{2}{19}$, $\theta = 208.1°$

9 Acceleration given, find when P returns to O (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t = 0, P passes through the fixed point O with velocity 96 m s^{-1}. At time t seconds, P has acceleration $(6t - 44)$ m s^{-2}.
Find (a) the times at which P is at O again, (b) the greatest distance of P from O before it first returns to O.

$v = \int a\,dt = 3t^2 - 44t + 96$ ($v = 96$ when $t = 0$)
$s = \int v\,dt = t^3 - 22t^2 + 96t$ ($s = 0$ when $t = 0$)
$s = t(t - 6)(t - 16)$, so P is at O when $t = 6$ and $t = 16$
Greatest distance when $v = 0$: $3t^2 - 44t + 96 = 0$, so $t = \frac{8}{3}$ (the smaller root)
$|s| = \frac{8}{3} \times \frac{10}{3} \times \frac{40}{3}$

Answer: (a) t = 6, t = 16 (b) $\frac{3200}{27}$ (= 119) m

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$.
The point P has coordinates $(0, -4, -7)$.
Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 1 \\ -2 \\ -10 \end{pmatrix} - t\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$
$\mathrm{FP} \cdot \mathbf{d} = 0$: $-26 - t(13) = 0$, so $t = -2$
F = $(-1, 2, -3)$
$\mathrm{PF} = \begin{pmatrix} 1 \\ -6 \\ -4 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{53}$

Answer: F = $(-1, 2, -3)$, PF = $\sqrt{53}$