Day 195 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + ax^2 + bx - 12$, where a and b are constants.
$(x + 4)$ is a factor of $f(x)$, and $f(x)$ leaves remainder -20 when divided by $(x - 1)$.
Find a and b, and hence solve $f(x) = 0$.

$f(-4) = 0$: $-64 + 16a - 4b - 12 = 0$, so $16a - 4b - 76 = 0$.
$f(1) = -20$: $a + b - 11 = -20$.
Solving: $a = 2$, $b = -11$.
$f(x) = x^3 + 2x^2 - 11x - 12 = (x + 4)(x + 1)(x - 3)$
Answer: a = 2, b = −11; x = −4, −1, 3

2 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are modelled by $N(27, \sigma^2)$. The probability of a value below 23 is 0.15.
(a) Find σ.
(b) Find the probability of a value above 32.

$P(X < 23) = 0.15$ so $Z = \frac{23 - 27}{\sigma} = -1.0364$

$\sigma = \frac{-4}{-1.0364} = 3.859$

$z = \frac{32 - 27}{3.859} = 1.296$

$P(X > 32) = P(Z > 1.296) = 0.0976$
Answer: (a) 3.86 (b) 0.0976

3 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 40\,e^{-x} = 13$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 13e^x + 40 = 0$
$(e^x - 5)(e^x - 8) = 0$
$e^x = 5$ or $e^x = 8$
Answer: x = ln 5 and x = ln 8

4 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 2 green counters and some black counters, n in total.
Two are taken without replacement. P(at least one green) = $\frac{13}{28}$. Find n.

P(neither green) = $\frac{(n-2)}{n} \times \frac{(n-3)}{(n-1)} = \frac{15}{28}$

$\frac{15}{28}\,n(n-1) = (n-2)(n-3)$
Rearranging gives a quadratic in n whose roots are $n = 8$ and one other value
The other root is $\frac{21}{13}$ so is rejected (not a valid number of counters)
$n = 8$
Answer: n = 8

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5 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{2}{19 + 12\sin\theta + 5\cos\theta}$

(a) Write $12\sin\theta + 5\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{12^2 + 5^2} = 13$, $\tan\alpha = \frac{5}{12}$, $\alpha = 22.6°$

Denominator $= 19 + 13\sin(\theta + 22.6°)$, which lies between 6 and 32

Greatest $f(\theta) = \frac{2}{6} = \frac{1}{3}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{2}{32} = \frac{1}{16}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 22.6° = 270°$, so $\theta = 247.4°$

Answer: (a) $13\sin(\theta + 22.6°)$ (b) greatest $\frac{1}{3}$, least $\frac{1}{16}$, $\theta = 247.4°$

6 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time $t = 0$ hours, A is at $(10\mathbf{i})$ km relative to a fixed origin and B is at $(-3\mathbf{i} - 16\mathbf{j})$ km. A has velocity $(-2\mathbf{i} + 2\mathbf{j})$ km h^{-1} and B has velocity $(2\mathbf{i} + 5\mathbf{j})$ km h^{-1}.

Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-13\mathbf{i} - 16\mathbf{j}) + t(4\mathbf{i} + 3\mathbf{j})$

$|\mathbf{d}|^2 = (4t - 13)^2 + (3t - 16)^2 = 25t^2 - 200t + 425$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-100}{25} = 4$

$\mathbf{d} = (-13\mathbf{i} - 16\mathbf{j}) + 4(4\mathbf{i} + 3\mathbf{j}) = (3\mathbf{i} - 4\mathbf{j})$, so least distance = 5

Answer: (a) $t = 4$ h (b) 5 km

7 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(27 - x)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^3.

State the values of x for which it is valid.

$(27 - x)^{\frac{1}{3}} = 27^{\frac{1}{3}}(1 - \frac{1}{27}x)^{\frac{1}{3}} = 3(1 + u)^{\frac{1}{3}}$, $u = -\frac{1}{27}x$.

$(1 + u)^{\frac{1}{3}} = 1 + \frac{1}{3}u + (-\frac{1}{9})u^2 + \frac{5}{81}u^3 + \ldots$

Substitute u and multiply by 3.

Valid when $|u| < 1$, i.e. $|x| < 27$.

Answer: $3 - \frac{1}{27}x - \frac{1}{2187}x^2 - \frac{5}{531441}x^3 + \ldots$; $|x| < 27$

8 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.

Line l_2 is perpendicular to l_1 at B and meets the x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{2}{2 + 2} = \frac{1}{2}$.

Gradient of $l_2 = -2$. Equation: $y - 2 = -2(x - 2)$.

$y = 0$: $x = 3$, so C(3, 0).

$AB = 2\sqrt{5}$, $BC = \sqrt{5}$: area = ½ × AB × BC.

Answer: C(3, 0); area 5

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9 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where $v = 72\sqrt{t} - 12t$.

Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{72}{2\sqrt{t}} - 12 = 0$ when $\sqrt{t} = 3$, so $t = 9$

$v = 72 \times 3 - 12 \times 9 = 108$

$v = 0$ when $\sqrt{t}(72 - 12\sqrt{t}) = 0$: $t = 36$ ($t = 0$ is the start)

$s = \int v\,dt = \frac{2}{3} \times 72\, t^{\frac{3}{2}} - \frac{12}{2}t^2$, so at $t = 36$: $s = 10368 - 7776$

Answer: (a) 108 m s^{-1} (b) 2592 m

10 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $8\text{cosec}^2\theta - 6\cot\theta - 10 = 0$

for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $8\cot^2\theta - 6\cot\theta - 2 = 0$

$2(4\cot\theta + 1)(\cot\theta - 1) = 0$

$\cot\theta = -\frac{1}{4}$ or $\cot\theta = 1$

$\tan\theta = -4$ or 1, so $\theta = 104.0°$ and $45.0°$ ($+180°$)

Answer: θ = 45.0°, 104.0°, 225.0°, 284.0°