Day 194 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coefficient of x^3 in the expansion of
$(3x - 3)(1 - 2x)^5$.

$(1 - 2x)^5$: x^3 term has coefficient $C(5,3) \times (-2)^3 = -80$, x^2 term has coefficient $C(5,2) \times (-2)^2 = 40$.
x^3 in the product: $(-3) \times (-80) + 3 \times 40$.
$= 360$.
Answer: 360

2 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos(2\theta - 45°) = -\frac{1}{2}$
for $0° \leq \theta \leq 360°$.

Let $u = 2\theta - 45°$; then $\cos u = -\frac{1}{2}$
Base solutions: $u = 120, -120°$ (+ multiples of 360°)
$\theta = (u + 45°) \div 2$, keeping those in range
Answer: θ = 82.5°, 142.5°, 262.5°, 322.5°

3 Grouped median with an unknown frequency (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The table shows the times, t minutes, taken by some runners.
Using linear interpolation, the median is estimated as 50 minutes. Find the value of k.

Total = 40 + k, so the median class is 40 to 60

$$\text{Median} = 40 + \frac{\frac{(40 + k)}{2} - (10 + k)}{11} \times 20 = 50$$

Simplify: $10 - k/2 = 5.5$
$k = 9$
Answer: k = 9

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $6\text{cosec}^2\theta - 2\cot\theta - 10 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $6\cot^2\theta - 2\cot\theta - 4 = 0$
$2(3\cot\theta + 2)(\cot\theta - 1) = 0$
$\cot\theta = -\frac{2}{3}$ or $\cot\theta = 1$
$\tan\theta = -\frac{3}{2}$ or 1, so $\theta = 123.7°$ and 45.0° (+180°)
Answer: θ = 45.0°, 123.7°, 225.0°, 303.7°

5 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the top of a vertical cliff, 24.5 m high, at 24.5 m s^{-1} at angle α above the horizontal ($\tan\alpha = \frac{4}{3}$), and lands on horizontal ground.
Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m s^{-2}.

Vertically (up positive): $-24.5 = 19.6\,t - 4.9t^2$, so $4.9t^2 - 19.6t - 24.5 = 0$
$t = 5$ s (the positive root)
Horizontal distance = $14.7 \times 5 = 73.5$ m
On landing: $v_x = 14.7$, $v_y = 19.6 - 9.8 \times 5 = -29.4$
speed = $\sqrt{(14.7^2 + 29.4^2)} = 32.9$, $\tan\text{angle} = \frac{29.4}{14.7}$, angle = 63.4° below the horizontal
Answer: (a) 5 s, 73.5 m (b) 32.9 m s^{-1} at 63.4° below the horizontal

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6 Poisson: scaling the rate, then binomial (CIE)

Emails arrive in an inbox at an average rate of 4 per hour, independently and at random.
(a) Find the probability of at least 3 in a 30-minute period.
(b) Find the probability that, of 8 separate 30-minute periods, exactly 2 have no arrivals.

30-minute period: $\lambda = 4 \times \frac{30}{60} = 2$, so $X \sim Po(2)$

(a) $P(X \geq 3) = 1 - P(X \leq 2) = 0.3233$

(b) $P(X = 0) = e^{-2} = 0.1353$

$Y \sim B(8, 0.1353)$: $P(Y = 2) = 0.214$

Answer: (a) 0.323 (b) 0.214

7 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 11}{x - 4}$ for $x \neq 4$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 4) = kx + 11$, so $x(y - k) = 4y + 11$.

$f^{-1}(x) = \frac{4x + 11}{x - k}$.

$f = f^{-1}$ requires $k = 4$.

$f(x) = x$: $4x + 11 = x(x - 4)$, so $x^2 - 8x - 11 = 0$.

$x = 4 \pm \sqrt{27} = 4 \pm 3\sqrt{3}$.

Answer: $k = 4$; $x = 4 \pm 3\sqrt{3}$

8 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through A(−5, 11), B(3, 15)
and C(13, 5). Find its equation.

Centre is on the perpendicular bisector of each chord.
Bisector of AB: $8x + 4y = 44$.
Bisector of BC: $10x - 10y = -20$.
Solving: centre (3, 5).
$r^2 = (-5 - 3)^2 + (11 - 5)^2 = 100$.

Answer: $(x - 3)^2 + (y - 5)^2 = 100$

9 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 10 m and mass 50 kg rests on supports at C and D, where AC = 3 m and DB = 3 m. A girl of mass 40 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $50(2) = 40(x - 7)$, so $x = 7 + \frac{50 \times 2}{40} = 9.5$

Tilting about C (girl before C): $50(2) = 40(3 - x)$, so $x = 3 - \frac{50 \times 2}{40} = 0.5$

Answer: $0.5 \leq x \leq 9.5$

10 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{24 + 8\sin\theta + 15\cos\theta}$

(a) Write $8\sin\theta + 15\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{8^2 + 15^2} = 17$, $\tan\alpha = \frac{15}{8}$, $\alpha = 61.9°$

Denominator $= 24 + 17\sin(\theta + 61.9°)$, which lies between 7 and 41

Greatest $f(\theta) = \frac{4}{7} = \frac{4}{7}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{41} = \frac{4}{41}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 61.9° = 270°$, so $\theta = 208.1°$

Answer: (a) $17\sin(\theta + 61.9°)$ (b) greatest $\frac{4}{7}$, least $\frac{4}{41}$, $\theta = 208.1°$