Day 193 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $9\sin\theta + 12\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $9\sin\theta + 12\cos\theta = 10$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{9^2 + 12^2} = 15$ and $\tan\alpha = \frac{12}{9}$, so $\alpha = 53.1°$

So $9\sin\theta + 12\cos\theta \equiv 15\sin(\theta + 53.1°)$

$\sin(\theta + \alpha) = \frac{10}{15}$

$\theta + \alpha = 41.8°$ or $138.2°$ (using $\alpha = 53.1°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) $R = 15$, $\alpha = 53.1°$ (b) $\theta = 85.1°, 348.7°$

2 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{4x - 2}{x + 2} \leq 3$.

Rearrange: $\frac{4x - 2}{x + 2} - 3 \leq 0$.

Combine: $\frac{x - 8}{x + 2} \leq 0$.

Critical values: $x = 8$ (zero of numerator), $x = -2$ (excluded, denominator zero).

Sign analysis of the three regions gives the solution.

Answer: $-2 < x \leq 8$

3 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(10, \frac{3}{10})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{10 - r}{r + 1} \times \frac{3}{7}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(10, r + 1)}{C(10, r)} \times \frac{p}{q} = \frac{10 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{3}{10}$ and $q = \frac{7}{10}$, $p \div q = \frac{3}{7}$

The ratio exceeds 1 when $(10 - r)\frac{3}{10} > (r + 1)\frac{7}{10}$, that is $r < 2.3$

So probabilities rise up to $r = 3$ and then fall; the mode is 3

$P(X = 3) = 0.267$

Answer: 3 with probability 0.267

4 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = x^3e^{-2x}$ for $x > 0$,
and determine its nature.

$dy/dx = x^2e^{-2x}(3 - 2x) = 0$ gives $x = 0$ or $x = \frac{3}{2}$

$y = \frac{27}{8}e^{-3}$ when $x = \frac{3}{2}$

Second derivative at this point is negative, so it is a maximum

Answer: $(\frac{3}{2}, \frac{27}{8}e^{-3})$, maximum

5 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 8 kg is at rest on rough horizontal ground ($\mu = 0.5$). A horizontal force of 50.4 N acts on it for 7 s and is then removed.

Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take $g = 9.8$ m s^{-2}.

Friction = $\mu mg = 0.5 \times 8 \times 9.8 = 39.2$ N

While pushed: $50.4 - 39.2 = 8a$, so $a = 1.4$

$v = 1.4 \times 7 = 9.8$; $s_1 = \frac{1}{2} \times 1.4 \times 7^2 = 34.3$

After release: deceleration = $\mu g = 4.9$, so $s_2 = \frac{9.8^2}{2 \times 4.9} = 9.8$

Total distance = 34.3 + 9.8

Answer: (a) 9.8 m s^{-1} (b) 44.1 m

6 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -8 \\ -2 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$.

The point P has coordinates (0, 1, −1).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} - t\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

FP · d = 0: $26 - t(13) = 0$, so $t = 2$

F = (−2, −2, 2)

$\mathrm{PF} = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{22}$

Answer: F = (−2, −2, 2), PF = $\sqrt{22}$

7 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 8 m and mass 12 kg rests on supports at C and D, where AC = 2 m and AD = 5 m, with $R_C = 0.5 R_D$.

Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 0.5 R_D$. Moments about A: $R_C \times 2 + R_D \times 5 = 12g \times \mathrm{AG}$

$R_D(0.5 \times 2 + 5) = 12g \times \mathrm{AG}$ with $R_D(1 + 0.5) = 12g$, so $\mathrm{AG} = \frac{0.5 \times 2 + 5}{1.5} = 4$

Tilting about D: $R_C = 0$. Moments about D: $mg(8 - 5) = 12g(5 - 4)$

$m = \frac{12 \times 1}{3} = 4$

Answer: (a) 4 m (b) 4

8 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(9, 8) lies on the circle: $(9 - r)^2 + (8 - r)^2 = r^2$.

$r^2 - 34r + 145 = 0$, so $r = 5$ or $r = 29$.

The diagram shows P on the larger circle, so $r = 29$.

$(x - 29)^2 + (y - 29)^2 = 841$, then expand.

Answer: $x^2 + y^2 - 58x - 58y + 841 = 0$

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9 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: $p = 0.2$ against H_1: $p < 0.2$ uses a sample of 25 at the 5% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact $p = 0.1$.

$X \sim B(25, 0.2)$. $P(X \leq 1) = 0.0274$ and $P(X \leq 2) = 0.0982$

The first is at most 0.05 and the second is not, so the critical region is $X \leq 1$

Actual significance level = $P(X \leq 1) = 0.0274$

Type II error: H_0 is not rejected, so $X \geq 2$, with $X \sim B(25, 0.1)$

$P(X \geq 2) = 1 - P(X \leq 1) = 0.729$

Answer: (a) $X \leq 1$ (b) 0.0274 (c) 0.729

10 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{2x - 3}{x + 3}$.

(a) Write down the equations of the asymptotes of C.

(b) The line $y = x + k$ is a tangent to C.

Find the possible values of k.

(a) Vertical: $x = -3$. As $x \to \infty$, $y \to 2$, so $y = 2$.

(b) $2x - 3 = (x + k)(x + 3)$.

$x^2 + (k + 1)x + (3k + 3) = 0$.

Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 - 10k - 11 = 0$.

Answer: (a) $x = -3$, $y = 2$ (b) $k = -1$ or $k = 11$