Day 192 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 14.7^2 - 2 \times 9.8 \times H$, so H = 11.025

At height $\frac{H}{2}$ = 5.5125: $v_y^2 = 14.7^2 - 2 \times 9.8 \times 5.5125 = 108.045$

v_x = 19.6 (constant), so speed^2 = 384.16 + 108.045 = 492.205

Answer: (a) 11.025 m (b) 22.2 m s^{-1}

2 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x + 1) the remainder is 4; when divided by (x − 2) the remainder is −5. Find the remainder when f(x) is divided by (x + 1)(x − 2).

$f(x) = (x + 1)(x - 2)Q(x) + px + q$.

$f(-1) = 4$: $-p + q = 4$.

$f(2) = -5$: $2p + q = -5$.

Subtracting: $-3p = 9$, so $p = -3$, $q = 1$.

Answer: $-3x + 1$

3 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 19 and upper quartile 29. An outlier is a value more than 1 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 8 observations have mean 23. The largest is 50, an outlier, and it is removed. Find the new mean.

IQR = 29 − 19 = 10, so 1 × IQR = 10

Lower limit = 19 − 10 = 9, upper limit = 29 + 10 = 39

$\Sigma x = 8 \times 23 = 184$, so without 50: $\Sigma x = 134$

New mean = $\frac{134}{7}$

Answer: (a) 9 to 39 (b) 19.14

4 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = -x - 3$ and $g(x) = x^2 - 4$.

Solve fg(x) = gf(x).

$fg(x) = -(x^2 - 4) - 3 = -x^2 + 1$.

$gf(x) = (-x - 3)^2 - 4 = x^2 + 6x + 5$.

Equate: $2x^2 + 6x + 4 = 0$.

Solve the quadratic: $x = -2$ or $x = -1$.

Answer: $x = -2$ or $x = -1$

5 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2t^2 - t$, $y = 2t^3 - 3t$.

Find the equation of the tangent at the point where t = 2, in the form y = mx + c.

$dx/dt = 4t - 1$, $dy/dt = 6t^2 - 3$

At t = 2: $dy/dx = \frac{21}{7} = 3$; point (6, 10)

$y - 10 = 3(x - 6)$

Answer: $y = 3x - 8$

6 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(28\mathbf{i} + 60\mathbf{j})$ km relative to a port O, and moves with constant velocity $(\mathbf{i})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 17 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(28\mathbf{i} + 60\mathbf{j}) + t(\mathbf{i})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 17 t, so $|(28\mathbf{i} + 60\mathbf{j}) + t(\mathbf{i})| = 17\,t$
Squaring and rearranging: $-288t^2 + 56t + 4384 = 0$, which gives t = 4 or a negative value
t = 4 (the other root is negative since B is faster than A)

A is then at $(28\mathbf{i} + 60\mathbf{j}) + 4(\mathbf{i}) = (32\mathbf{i} + 60\mathbf{j})$, so $v_B = \frac{1}{4}(32\mathbf{i} + 60\mathbf{j}) = (8\mathbf{i} + 15\mathbf{j})$

Answer: (a) 4 h (b) $(8\mathbf{i} + 15\mathbf{j})$ km h^{-1}

7 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(16, \frac{1}{5})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{16 - r}{r + 1} \times \frac{1}{4}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(16, r + 1)}{C(16, r)} \times \frac{p}{q} = \frac{16 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{5}$ and $q = \frac{4}{5}$, $p \div q = \frac{1}{4}$

The ratio exceeds 1 when $(16 - r)\frac{1}{5} > (r + 1)\frac{4}{5}$, that is $r < 2.4$

So probabilities rise up to r = 3 and then fall; the mode is 3
P(X = 3) = 0.246
Answer: 3 with probability 0.246

8 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = x + 4 to find the exact value of

$\int_0^5 \frac{x}{\sqrt{x + 4}}\,dx$

$x = u - 4$, $dx = du$; limits become u = 4 and u = 9

Integral $= \int_4^9 \frac{u - 4}{\sqrt{u}}\,du = [\frac{2}{3}u^{3/2} - 8u^{1/2}]$

Evaluate the limits

Answer: $\frac{14}{3}$

9 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{13x - 5}{(1 - 3x)(1 - x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $13x - 5 \equiv A(1 - x) + B(1 - 3x)$: $A = -1$, $B = -4$.

$f(x) = \frac{-1}{1 - 3x} - \frac{4}{1 - x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use c = −3, then c = −1).

Coefficients of x^j: $-1(3)^j - 4(1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = -1$, $B = -4$ (b) $-5 - 7x - 13x^2 - 31x^3 + \ldots$; $|x| < \frac{1}{3}$

10 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 4 \sin t$, $y = 3 \cos t$, for $0 \leq t \leq \frac{\pi}{2}$. Find the exact area bounded by the curve and the coordinate axes.

$dx/dt = 4 \cos t$

Area $= \int_0^{\pi/2} 3 \cos t \times 4 \cos t \, dt = 12 \int_0^{\pi/2} \cos^2 t \, dt$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$: integral $= 12 \times \frac{\pi}{4}$

Answer: 3π