Day 190 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 10.8 m, at a horizontal distance of 27 m from O.

Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8$ m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$10.8 = 27 - \frac{9.8 \times 27^2}{u^2}$, so $u^2 = \frac{9.8 \times 729}{16.2} = 441$

$u = 21$

Greatest height $= \frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{441}{39.2}$

Answer: (a) 21 m s^{-1} (b) 11.2 m

2 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coefficient of x^2 in the expansion of

$(x + 3)(1 + 2x)^8$.

$(1 + 2x)^8$: x^2 term has coefficient $C(8,2) \times 2^2 = 112$, x^1 term has coefficient $C(8,1) \times 2^1 = 16$.

x^2 in the product: $3 \times 112 + 1 \times 16$.

$= 352$.

Answer: 352

3 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $e^x = 5x + 2$ has a root α.

(a) Show that the equation can be written as $x = \ln(5x + 2)$.

(b) Use $x_{n+1} = \ln(5x_n + 2)$ with $x_0 = 2$ to find x_1, x_2, x_3

to 4 d.p., and hence state α to 3 d.p.

Take natural logarithms of both sides to obtain $x = \ln(5x + 2)$

$x_1 = 2.4849$, $x_2 = 2.6689$, $x_3 = 2.7308$

The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.760$

Answer: $x_1 = 2.4849$, $x_2 = 2.6689$, $x_3 = 2.7308$; $\alpha = 2.760$

4 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $5^{2x} - 6 \times 5^x + 5 = 0$.

Let $y = 5^x$. Then $y^2 - 6y + 5 = 0$.

$(y - 1)(y - 5) = 0$, so $y = 1$ or $y = 5$.

$5^x = 1$ gives $x = 0$; $5^x = 5$ gives $x = 1$.

Answer: $x = 0$ or $x = 1$

5 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + 5x - 4$.

(a) Show that $f(x) = 0$ has a root α with $0 < \alpha < 1$.

(b) Use Newton-Raphson with $x_0 = 0$ to find x_1 and x_2

to 4 decimal places.

$f(0) = -4$ and $f(1) = 2$: a change of sign, f continuous

$f'(x) = 3x^2 + 5$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 0 - \frac{-4.000}{5.000} = 0.8000$

$x_2 = 0.8000 - \frac{0.5120}{6.920} = 0.7260$

Answer: $x_1 = 0.8000$, $x_2 = 0.7260$

6 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{24}{x^2 + 8x + 20}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 + 8x + 20 = (x + 4)^2 + 4$.

$(x + 4)^2 \geq 0$, so the denominator is at least 4, with equality at $x = -4$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{24}{4} = 6$ at $x = -4$.

Answer: greatest value 6 at $x = -4$; range $0 < f(x) \leq 6$

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−2, 0, −1), (5, −3, 3), (−3, −1, −3).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} -7 \\ 3 \\ -4 \end{pmatrix}$, $BC = \begin{pmatrix} -8 \\ 2 \\ -6 \end{pmatrix}$; $BA \cdot BC = 86$

$|BA| = \sqrt{74}$, $|BC| = \sqrt{104}$

$\cos ABC = \frac{86}{\sqrt{74}\sqrt{104}} = 0.9803$, so $ABC = 11.4°$

$|BA \times BC| = \left|\begin{pmatrix} -10 \\ -10 \\ 10 \end{pmatrix}\right| = \sqrt{300}$

$\text{Area} = \frac{1}{2}|BA \times BC|$

Answer: ABC = 11.4°, area = $5\sqrt{3}$

8 Pdf of the form a + bx with a given mean (CIE)

The pdf of X is $f(x) = a + bx$ for $0 \leq x \leq 4$, and 0 otherwise, where a and b are constants. $E(X) = \frac{38}{15}$.

(a) Find a and b.

(b) Find P(X < 2).

Total area = 1: $4a + 8b = 1$

$E(X) = \int x(a + bx)\,dx = 8a + \frac{64}{3}b = \frac{38}{15}$

Solving the simultaneous equations: $a = \frac{1}{20}$, $b = \frac{1}{10}$

$P(X < 2) = \int f(x)\,dx$ from 0 to $2 = \frac{1}{20} \times 2 + \frac{1}{10} \times 2$

$= \frac{3}{10}$

Answer: (a) $a = \frac{1}{20}$, $b = \frac{1}{10}$ (b) $\frac{3}{10}$

9 Slide down a rough slope then along rough ground (CIE)

A particle starts from rest at the top of a rough slope of length 6 m ($\sin\alpha = \frac{4}{5}$, $\mu = 0.6$), then slides along rough horizontal ground with the same μ.

Use energy to find (a) its speed at the foot of the slope (3 s.f.), (b) the distance it slides on the ground. Take $g = 9.8\ \text{m s}^{-2}$.

On the slope: $\frac{1}{2} m v^2 = m g \times 6 \sin\alpha - \mu m g \cos\alpha \times 6$

$v^2 = 2 \times 9.8 \times 6(\frac{4}{5} - 0.6 \times \frac{3}{5}) = 51.744$, so $v = 7.193$

On the ground: $\frac{1}{2} m v^2 = \mu m g x$, so $x = \frac{v^2}{2\mu g} = \frac{51.744}{11.76} = 4.4$

Answer: (a) 7.19 m s^{-1} (b) 4.4 m

10 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 0, 1, 2, 3 with $P(X = x) = k(x + 1)$. X_1 and X_2 are independent copies of X.

(a) Find k.

(b) Find $P(X_1 + X_2 = 2)$.

(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(1 + 2 + 3 + 4) = 10\ k = 1$, so $k = \frac{1}{10}$

(b) Ways to total 2: (0,2), (1,1), (2,0)

$P = \frac{1}{10} \times \frac{3}{10} + \frac{1}{5} \times \frac{1}{5} + \frac{3}{10} \times \frac{1}{10} = \frac{1}{10}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{3}{10}$

$E(X) = 2$, so $E(X_1 + X_2) = 2 \times 2 = 4$

Answer: (a) k = $\frac{1}{10}$ (b) $\frac{1}{10}$ (c) $\frac{3}{10}$, 4