Day 19 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class R has 30 students with mean mark 37 and standard deviation 2. Class S has 40 students with mean mark 74 and standard deviation 2.

(a) Find the mean mark of all 70 students.

(b) Find the standard deviation of all 70 students.

Σx for R = $30 \times 37 = 1110$, Σx for S = $40 \times 74 = 2960$

Combined mean = $\frac{4070}{70} = 58.1$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $30(4 + 1369) + 40(4 + 5476) = 260390$

Variance = $\frac{260390}{70} - 58.143^2 = 339.27$, so $\sigma = 18.42$

Answer: (a) 58.1 (b) 18.4

2 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{21 + 8\sqrt{5}}$ in the form

$a + b\sqrt{5}$, where a and b are positive integers.

Suppose $\sqrt{21 + 8\sqrt{5}} = a + b\sqrt{5}$. Square both sides.

$21 + 8\sqrt{5} = a^2 + 5b^2 + 2ab\sqrt{5}$.

Equate: $a^2 + 5b^2 = 21$ and $2ab = 8$, so $ab = 4$.

Try $a = 4$, $b = 1$: $16 + 5 = 21$ ☐.

($a > b\sqrt{5}$, so the root is positive.)

Answer: $4 + 1\sqrt{5}$

3 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 23 and upper quartile 40. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 12 observations have mean 35. The largest is 74, an outlier, and it is removed. Find the new mean.

IQR = $40 - 23 = 17$, so $1.5 \times \text{IQR} = 25.5$

Lower limit = $23 - 25.5 = -2.5$, upper limit = $40 + 25.5 = 65.5$

$\Sigma x = 12 \times 35 = 420$, so without 74: $\Sigma x = 346$

New mean = $\frac{346}{11}$

Answer: (a) -2.5 to 65.5 (b) 31.45

4 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $8 \sin\theta + 6 \cos\theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.

(b) Hence solve $8 \sin\theta + 6 \cos\theta = -2$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{8^2 + 6^2} = 10$ and $\tan\alpha = \frac{6}{8}$, so $\alpha = 36.9°$

So $8 \sin\theta + 6 \cos\theta \equiv 10 \sin(\theta + 36.9°)$

$\sin(\theta + \alpha) = \frac{-2}{10}$

$\theta + \alpha = -11.5°$ or $191.5°$ (using $\alpha = 36.9°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 10, α = 36.9° (b) θ = 154.7°, 311.6°

5 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car P passes a point O travelling at a constant speed of 16 m s^{-1}. 4 s later, car Q leaves O from rest with constant acceleration 2.5 m s^{-2}, travelling in the same direction.
Q overtakes P at the point X. Find (a) the time Q takes to reach X, (b) the distance OX, (c) the speed of Q at X.

Let Q travel for t s. P: distance = 16(t + 4); Q: distance = ½ × 2.5 × t^2
Equate: $2.5t^2 - 32t - 128 = 0$, so $(t - 16)(2.5t + 8) = 0$
t = 16 (the other root is negative)
OX = 2.5 × 16^2 / 2 = 320 m, and speed = 2.5 × 16 = 40
Answer: (a) 16 s (b) 320 m (c) 40 m s^{-1}

6 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 - 5x - 14 \geq 0$ and $x + 5 \geq 1$.

Quadratic: $(x + 2)(x - 7) \geq 0$ gives $x \leq -2$ or $x \geq 7$.
Linear: $x + 5 \geq 1$ gives $x \geq -4$.
Both must hold: take the overlap of the two sets.
Answer: $-4 \leq x \leq -2$ or $x \geq 7$

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For $0 \leq x \leq \pi$, the curves $y = 3 \sin x$ and $y = 3 \sin 2x$
meet at the origin and at $x = \frac{\pi}{3}$. Find the exact area
enclosed between them for $0 \leq x \leq \frac{\pi}{3}$.

For $0 < x < \frac{\pi}{3}$, $\sin 2x > \sin x$

Area $= 3\int_0^{\pi/3} (\sin 2x - \sin x)\,dx = 3[-\frac{\cos 2x}{2} + \cos x]$

$= 3[(-\cos\frac{2\pi}{3}/2 + \cos\frac{\pi}{3}) - (-\frac{1}{2} + 1)] = 3[\frac{3}{4} - \frac{1}{2}]$

Answer: $\frac{3}{4}$

8 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - xy + y^2 = 3$.
Find the coordinates of the points on C where the
tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -y}{-x + 2y}$

Horizontal tangent when $2x + -y = 0$, so $y = \frac{2x}{1}$

Substitute into the equation of C: $x^2 + (-xy) + (y^2) = 3$
Solve for the remaining coordinate
Answer: (−1, −2) and (1, 2)

9 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle has its centre in the first quadrant and touches both the x-axis and the y-axis. It passes through the point (8, 1).
Find the two possible equations of the circle.

Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$.
Substitute (8, 1): $(8 - r)^2 + (1 - r)^2 = r^2$.
$r^2 - 18r + 65 = 0$.
$(r - 5)(r - 13) = 0$, so r = 5 or r = 13.
Answer: $(x - 5)^2 + (y - 5)^2 = 25$ or $(x - 13)^2 + (y - 13)^2 = 169$

10 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 60 kg rests on supports at C and D, where AC = 3 m and DB = 3 m. A girl of mass 60 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $60(1) = 60(x - 5)$, so $x = 5 + \frac{60 \times 1}{60} = 6$

Tilting about C (girl before C): $60(1) = 60(3 - x)$, so $x = 3 - \frac{60 \times 1}{60} = 2$

Answer: $2 \leq x \leq 6$