Day 189 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3^x - 5x$ has one stationary point.
Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 3^x \ln 3 - 5 = 0$

$3^x = \frac{5}{\ln 3}$, $x = \frac{1}{\ln 3} \ln(\frac{5}{\ln 3}) = 1.38$

$d^2y/dx^2 = 3^x(\ln 3)^2 > 0$ for all x: minimum

$y = \frac{5}{\ln 3} - 5 \times x = -2.35$

Answer: (1.38, −2.35)

2 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1500 kg tows a trailer of mass 500 kg along a horizontal road, with driving force 2375 N. Resistances are 800 N on the car and 375 N on the trailer.
(a) Find the tension in the tow bar. At 7.5 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{2375 - 800 - 375}{2000} = 0.6$

Trailer: $T - 375 = 500 \times 0.6$, so $T = 675$

After the break the trailer decelerates at $\frac{375}{500} = 0.75$, and stops after $t = \frac{7.5}{0.75} = 10$ s

Trailer travels $\frac{7.5}{2} \times 10 = 37.5$ m

Car: $a = \frac{2375 - 800}{1500} = 1.05$, so $s = 7.5 \times 10 + ½ \times 1.05 \times 10^2 = 127.5$ m

Distance apart = 127.5 − 37.5

Answer: (a) 675 N (b) 90 m

3 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $24 \cos \theta + 7 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $24 \cos \theta + 7 \sin \theta = -8$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{24^2 + 7^2} = 25$ and $\tan \alpha = \frac{7}{24}$, so $\alpha = 16.3°$

So $24 \cos \theta + 7 \sin \theta \equiv 25 \cos(\theta - 16.3°)$

$\cos(\theta - \alpha) = \frac{-8}{25}$

$\theta - \alpha = \pm 108.7°$ (using $\alpha = 16.3°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 25, α = 16.3° (b) θ = 124.9°, 267.6°

4 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(7r + 1) = 2300$, find n.

$\sum_{r=1}^{n}(7r + 1) = 7 \times \frac{n(n + 1)}{2} + n.$

$\frac{7\,n(n + 1)}{2} + n = 2300$, so $7n^2 + 9n - 4600 = 0$.

Solving ($n > 0$): $n = 25$.

Answer: n = 25

5 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 24 and variance 6.
(a) Find n and p.
(b) Find P(X = 22), correct to 3 significant figures.

$np = 24$ and $np(1 - p) = 6$

Divide: $1 - p = 6 \div 24 = \frac{1}{4}$, so $p = \frac{3}{4}$

$n = 24 \div \frac{3}{4} = 32$

$P(X = 22) = C(32, 22) (\frac{3}{4})^{22} (\frac{1}{4})^{10} = 0.110$

Answer: (a) n = 32, p = $\frac{3}{4}$ (b) 0.110

6 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (3 kg) and B (3 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.1$, and B is 0.5 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.1 \times 3 \times 9.8 = 3a$. B: $3 \times 9.8 - T = 3a$
Adding: $(3 - 0.1 \times 3) \times 9.8 = 6a$, so $a = 4.41$
$v^2 = 2 \times 4.41 \times 0.5 = 4.41$, so $v = 2.1$

After B lands, A decelerates at $\mu g = 0.98$: extra distance = $\frac{2.1^2}{2 \times 0.98} = 2.25$

Total distance = 0.5 + 2.25
Answer: 2.75 m

7 Continuous random variable: pdf with unknown constant (CIE)

The pdf of X is $f(x) = \frac{k}{x^2}$ for $1 \le x \le 3$, and 0 otherwise.

(a) Find k.
(b) Find the median of X.
(c) Find E(X), to 3 significant figures.

(a) $\int k x^{-2} dx$ from 1 to 3 $= k(1 - \frac{1}{3}) = 1$, so $k = \frac{3}{2}$

(b) $\int f(x) dx$ from 1 to m $= \frac{3}{2}(1 - \frac{1}{m}) = \frac{1}{2}$

$1 - \frac{1}{m} = \frac{1}{3}$, so $m = \frac{3}{2}$

(c) $E(X) = \int \frac{3}{2} x^{-1} dx = \frac{3}{2} \ln 3 = 1.65$

Answer: (a) k = $\frac{3}{2}$ (b) $\frac{3}{2}$ (c) 1.65

8 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = t$, $y = 4t^2$. The line $y = -4x + 24$
meets C at A and B. Find the exact length AB.

Substitute: $4t^2 = -4t + 24$.
$4t^2 + 4t - 24 = 0 \Rightarrow t = -3$ or $t = 2$.
A(−3, 36), B(2, 16).
$AB^2 = (-5)^2 + 20^2 = 425$.
Answer: AB = $5\sqrt{17}$

9 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{3}{16 + 12\sin\theta + 5\cos\theta}$

(a) Write $12\sin\theta + 5\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{12^2 + 5^2} = 13$, $\tan\alpha = \frac{5}{12}$, $\alpha = 22.6°$

Denominator $= 16 + 13\sin(\theta + 22.6°)$, which lies between 3 and 29

Greatest $f(\theta) = \frac{3}{3} = 1$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{3}{29} = \frac{3}{29}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 22.6° = 270°$, so $\theta = 247.4°$

Answer: (a) $13\sin(\theta + 22.6°)$ (b) greatest 1, least $\frac{3}{29}$, $\theta = 247.4°$

10 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(−7, 6) and B(1, −10). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - (-7))^2 + 36 = (c - 1)^2 + 100$.

Expand and solve: $16c = 16$, so $c = 1$.

Area $= \frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$

$= \frac{1}{2}|(-8) \times (-10) - 0 \times 6| = 40$.

Answer: C(1, 0); area 40