Day 187 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A student's time on a task, in minutes, is N(42, 7^2).
Given that it exceeds 49, find the probability that it exceeds 55.

Required = $P(X > 55 \mid X > 49) = \frac{P(X > 55)}{P(X > 49)}$

$P(X > 55) = P(Z > 1.857) = 0.03165$

$P(X > 49) = P(Z > 1.000) = 0.1587$

$= \frac{0.03165}{0.1587} = 0.199$

Answer: 0.199

2 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_3 x - \log_3 y = 1$
$x + y = 12$

$\log_3(x/y) = 1$, so $x = 3^1 y = 3y$

$3y + y = 12$, so $4y = 12$

Answer: x = 9, y = 3

3 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 39.2^2 - 2 \times 9.8 \times H$, so $H = 78.4$

At height $\frac{H}{2} = 39.2$: $v_y^2 = 39.2^2 - 2 \times 9.8 \times 39.2 = 768.32$

$v_x = 29.4$ (constant), so $\text{speed}^2 = 864.36 + 768.32 = 1632.68$

Answer: (a) 78.4 m (b) 40.4 m s^{-1}

4 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $8\cos\theta - 6\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $8\cos\theta - 6\sin\theta = 7$ for $0 \le \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{8^2 + 6^2} = 10$ and $\tan\alpha = \frac{6}{8}$, so $\alpha = 36.9°$

So $8\cos\theta - 6\sin\theta \equiv 10\cos(\theta + 36.9°)$

$\cos(\theta + \alpha) = \frac{7}{10}$

$\theta + \alpha = \pm 45.6°$ (using $\alpha = 36.9°$)

Adding or subtracting multiples of 360° to give $0 \le \theta < 360°$

Answer: (a) R = 10, α = 36.9° (b) θ = 8.7°, 277.6°

5 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 - n + 17$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.

For n = 17: $17^2 - 17 + 17 = 289$.

$289 = 17 \times 17$, which is not prime.

All smaller values of n give a prime, so n = 17 is the smallest.

Answer: n = 17 (value 289 = 17 × 17)

6 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The breaking strength of a cable is N(μ, 64) with known variance 64. The mean was claimed to be 31 kN.
A random sample of 64 has mean 28.5. Test at the 5% level whether the mean is different from 31.

H_0: $\mu = 31$, H_1: $\mu \neq 31$. Under H_0, the sample mean ~ $N(31, \frac{64}{64})$

$z = \frac{28.5 - 31}{8 \div \sqrt{64}} = -2.500$

One tail: $P(Z < -2.500) = 0.00621$, compared with 2.5%

$0.00621 < 2.5\%$, so reject H_0

There is sufficient evidence at the 5% level that the mean has changed as claimed

Answer: Reject H_0 ($z = -2.500$, p-value 0.00621)

7 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = e^x$, the x-axis and the lines
$x = 0$ and $x = 1$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 (e^x)^2\,dx$

Simplify the integrand, then integrate

$V = \pi \times (-\frac{1}{2} + \frac{1}{2}e^2)$

Answer: $\frac{1}{2}\pi(-1 + e^2)$

8 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -3$ and

$u_{n+1} = \frac{3u_n + 4}{3u_n - 3}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{99} u_r$.

$u_1 = -3$, $u_2 = \frac{5}{12}$, $u_3 = -3$.

$u_3 = u_1$, so the sequence repeats with period 2.

One period sums to $-\frac{31}{12}$.

$99 = 2 \times 49 + 1$: 49 full periods plus the first 1 term.

Answer: $-\frac{1555}{12}$

9 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 4 kg moves under a single force F N, where $F = (24t - 12)\mathbf{i} + (36t^2 - 8)\mathbf{j}$ and t is the time in seconds.
At t = 0, P is at the origin with velocity $(4\mathbf{i} + 2\mathbf{j})$ m s^{-1}.
Find (a) the velocity of P at t = 2, (b) the position vector of P at t = 2.

$a = \frac{F}{4} = (6t - 3)\mathbf{i} + (9t^2 - 2)\mathbf{j}$

$v = \int a\,dt = (3t^2 - 3t + 4)\mathbf{i} + (3t^3 - 2t + 2)\mathbf{j}$ (using $v(0) = (4\mathbf{i} + 2\mathbf{j})$)
At t = 2: $v = (10\mathbf{i} + 22\mathbf{j})$
$r = \int v\,dt = (t^3 - 1.5t^2 + 4t)\mathbf{i} + (0.75t^4 - t^2 + 2t)\mathbf{j}$
At t = 2: $r = (10\mathbf{i} + 12\mathbf{j})$

Answer: (a) $(10\mathbf{i} + 22\mathbf{j})$ m s^{-1} (b) $(10\mathbf{i} + 12\mathbf{j})$ m

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10 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$4^x \times 16^y = 64$

$9^x \div 3^y = 3$

Write each side with prime bases: $64 = 2^6$, $3 = 3^1$.

First: $2^{2x} \times 2^{4y} = 2^6$, so $2x + 4y = 6$.

Second: $3^{2x - 1y} = 3^1$, so $2x - 1y = 1$.

Solving the pair of linear equations: $x = 1$, $y = 1$.

Answer: x = 1, y = 1