**Day 184** **A Level Maths: S Tier 10-a-day: Model Solutions**

## 1 Dropped and projected particles meet (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is projected vertically upwards from ground level with speed 23 m $s^{-1}$. At the same instant, ball B is released from rest 23 m directly above A.
The balls collide. Find (a) the height above the ground at which they collide, (b) the velocity of A just before the collision.
Take g = 9.8 m $s^{-2}$.

Relative to B, A moves upwards at the constant speed 23 m $s^{-1}$ (both have acceleration g)
So the balls meet when 23 t = 23, giving t = 1
Height of A: $s = 23 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 18.1$
$v = u - gt = 23 - 9.8 \times 1 = 13.2$
**Answer: (a) 18.1 m (b) 13.2 m $s^{-1}$ upwards**

## 2 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 8 strips to estimate
$\int_1^2 \frac{\ln x}{x}\,dx$, to 4 s.f.
Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{8}$; ordinates at $x = 1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \ldots, 2$

$\text{Sum} = \frac{h}{2}[y_0 + y_8 + 2(y_1 + \ldots + y_7)] = 0.2390$

f''(x) < 0 on the interval, so the curve is concave and the trapezia lie below the curve
**Answer: 0.2390; underestimate**

## 3 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2 - x + 4$ is translated by $\begin{pmatrix}3\\1\end{pmatrix}$
and then a stretch, scale factor 3, parallel to the y-axis.
Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix}3\\1\end{pmatrix}$: $y = f(x - 3) + 1 = 2x^2 - 13x + 26$.

Then a stretch, scale factor 3, parallel to the y-axis: multiply y by 3.
Expand and simplify to give the quadratic.
**Answer: $y = 6x^2 - 39x + 78$**

## 4 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

P(X = x) = k(x + 2) for x = 1, 2, 3, and 0 otherwise.
(a) Show that $k = \frac{1}{12}$.
(b) Find E(X) and Var(X).
(c) Find Var(4X + 3).

(a) $\Sigma\, P(X = x) = k(3 + 4 + 5) = 12\,k = 1$, so $k = \frac{1}{12}$

(b) $E(X) = \Sigma\, x\, P(X = x) = \frac{1\times3 + 2\times4 + 3\times5}{12} = \frac{13}{6}$

$E(X^2) = \frac{1\times3 + 4\times4 + 9\times5}{12} = \frac{16}{3}$

$\text{Var}(X) = \frac{16}{3} - \left(\frac{13}{6}\right)^2 = \frac{23}{36}$

(c) $\text{Var}(4X + 3) = 4^2\,\text{Var}(X) = 16 \times \frac{23}{36} = \frac{92}{9}$

**Answer: (b) $E(X) = \frac{13}{6}$, $\text{Var}(X) = \frac{23}{36}$ (c) $\frac{92}{9}$**

## 5 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 250 kg, carrying a passenger of mass 70 kg, is supported by a vertical cable. The lift is moving upwards and decelerating at 1.2 m $s^{-2}$.

Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m $s^{-2}$.

Take the direction of the acceleration as positive: it is downwards

Passenger: $70 \times 9.8 - R = 70 \times 1.2$

$R = 70 \times (9.8 - 1.2) = 602$

Lift and passenger: $320 \times 9.8 - T = (250 + 70) \times 1.2$

$T = 320 \times 8.6 = 2752$

**Answer: (a) 602 N (b) 2752 N**

## 6 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The volume of a sphere increases at a constant rate of $36\pi$ $cm^3s^{-1}$.

When the radius is 3 cm, find the rate of increase of (a) the radius,

(b) the surface area. Volume = $\frac{4}{3}\pi r^3$.

$dV/dt = 4\pi r^2\, dr/dt$, so $36\pi = 4\pi \times 9 \times dr/dt$

$dr/dt = 1$

$A = 4\pi r^2$, so $dA/dt = 8\pi r\, dr/dt = 8\pi \times 3 \times 1$

**Answer: (a) 1 cm $s^{-1}$ (b) $24\pi$ $cm^2s^{-1}$**

## 7 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = x^2 - 6x + 10$

for $x \leq k$. Given that k is the greatest value for which

f is one-one, find k, $f^{-1}(x)$, and the domain of $f^{-1}$.

Complete the square: $f(x) = (x - 3)^2 + 1$.

The vertex is at x = 3, so k = 3 (f is one-one on one side of it).

Let y = f(x): $y - 1 = (x - 3)^2$, so $(x - 3) = \pm\sqrt{y - 1}$.

As $x \leq 3$, take the negative root: $x = 3 - \sqrt{y - 1}$.

Range of f = domain of $f^{-1}$: f(3) = 1, so $x \geq 1$.

**Answer: $k = 3$; $f^{-1}(x) = 3 - \sqrt{x - 1}$; $x \geq 1$**

## 8 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations

$x = 2 + 4\cos t$, $y = 4 + 4\sin t$.

Find its Cartesian equation.

$\cos t = \frac{x - 2}{4}$, $\sin t = \frac{y - 4}{4}$.

Use $\cos^2 t + \sin^2 t = 1$:

$(x - 2)^2 + (y - 4)^2 = 16$.

**Answer: $(x - 2)^2 + (y - 4)^2 = 16$**

## 9 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = \sqrt{x}e^x$, the x-axis and the lines

$x = 0$ and $x = 1$ is rotated through $2\pi$ about the x-axis.

Find the exact volume of the solid formed.

$V = \pi\int_0^1 y^2\, dx = \pi\int_0^1 (\sqrt{x}e^x)^2\, dx$

Simplify the integrand, then integrate

$V = \pi \times (\frac{1}{4} + \frac{1}{4}e^2)$

**Answer: $\frac{1}{4}\pi(1 + e^2)$**

## 10 Poisson hypothesis test (CIE)

Faults in a cable occur at random at a mean rate of 3 per day. After a modification, 8 faults are found in 1 day. Test at the 10% level whether the rate of faults has increased.

$H_0$: $\lambda = 3$ per day, $H_1$: $\lambda > 3$ per day

Over 1 day: $X \sim Po(3)$

$P(X \geq 8) = 1 - P(X \leq 7) = 0.0119$

$0.0119 < 10\%$, so reject $H_0$

There is sufficient evidence that the rate has increased

**Answer: Reject $H_0$ (p-value 0.0119)**