Day 182 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 6x^2 + 17x - 24$.

Show that $(x - 3)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(3) = 27 - 54 + 51 - 24 = 0$, so $(x - 3)$ is a factor.

$f(x) = (x - 3)(x^2 - 3x + 8)$.

For $x^2 - 3x + 8 = 0$: $b^2 - 4ac = 9 - 32 = -23 < 0$.

No real roots from the quadratic, so the only real root is $x = 3$.

Answer: Proved; the only real root is $x = 3$

2 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Find the horizontal distance between the two points at which the tennis ball is 73.5 m above the ground.

Take $g = 9.8$ m s^{-2}.

Vertical: $73.5 = 39.2\,t - 4.9t^2$, i.e. $4.9t^2 - 39.2t + 73.5 = 0$

$$t = \frac{39.2 \pm \sqrt{(39.2^2 - 19.6 \times 73.5)}}{9.8} = \frac{39.2 \pm 9.8}{9.8}$$

The two times differ by 2 s (symmetry about the top of the path)

Horizontal distance = 29.4×2

Answer: 58.8 m

3 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$$f(\theta) = \frac{1}{20 + 12\sin\theta + 5\cos\theta}$$

(a) Write $12\sin\theta + 5\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{12^2 + 5^2} = 13$, $\tan\alpha = \frac{5}{12}$, $\alpha = 22.6°$

Denominator $= 20 + 13\sin(\theta + 22.6°)$, which lies between 7 and 33

Greatest $f(\theta) = \frac{1}{7} = \frac{1}{7}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{1}{33} = \frac{1}{33}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 22.6° = 270°$, so $\theta = 247.4°$

Answer: (a) $13\sin(\theta + 22.6°)$ (b) greatest $\frac{1}{7}$, least $\frac{1}{33}$, $\theta = 247.4°$

4 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^2 + 2$ is never a multiple of 4, for any integer n.

Every integer is $4q$, $4q + 1$, $4q + 2$, $4q + 3$ for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 4$, so:

$0 \to 0$; $1 \to 1$; $2 \to 0$; $3 \to 1$ ($r \to n^2 \bmod 4$)

So $n^2 + 2 \equiv 2$ or $3 \pmod 4$.

No residue is 0, so $n^2 + 2$ is never a multiple of 4.

Answer: Proved

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 10 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 100$, and after 6 minutes $\theta = 64$.
Find A and k, then find when $\theta = 44$ (3 s.f.).

$t = 0$: $10 + A = 100$ so $A = 90$

$t = 6$: $10 + 90e^{-6k} = 64$ so $e^{-6k} = \frac{3}{5}$

$k = \frac{1}{6} \ln \frac{5}{3} = 0.0851$

$90\, e^{-kt} = 34$ so $t = \frac{1}{k} \ln \frac{45}{17}$

$t = 11.4$ minutes

Answer: A = 90, k = 0.0851, t = 11.4 minutes

6 Symmetric limits of a normal distribution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(43, 49^2)$. Find the value k such that $P(43 - k < X < 43 + k) = 0.70$.
Give k to 3 significant figures.

The interval is symmetrical about the mean, so each tail has probability 0.150

$P(X < 43 - k) = 0.150$, so $\frac{-k}{7} = -1.0364$

$k = 7 \times 1.0364 = 7.255$

Answer: k = 7.26

7 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n - 4$, $u_1 = 7$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 533$.

$u_n = u_1 + \sum_{k=1}^{n-1} (k - 4) = 7 + 1 \times \frac{(n-1)n}{2} - 4(n-1)$.

$= \frac{1}{2}n^2 - \frac{9}{2}n + 11$.

So $a = \frac{1}{2}$, $b = -\frac{9}{2}$, $c = 11$.

$u_{37} = 529 \leq 533$ and $u_{38} = 562 > 533$.

Answer: (a) $a = \frac{1}{2}$, $b = -\frac{9}{2}$, $c = 11$ (b) n = 38

8 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ab^x$. The graph of lg y against x is the straight line shown.
Find a and b to 3 significant figures.

$\lg y = \lg a + x \lg b$. The intercept is $\lg a = 0.8$, so $a = 10^{0.8} = 6.310$

Gradient $= \frac{2.4 - 0.8}{8} = 0.2$

$\lg b = 0.2$ so $b = 10^{0.2} = 1.585$

Answer: a = 6.31, b = 1.58

9 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} -8 \\ -4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $-8 + 3s = 1$; $-4 + 3s = -1 - 3t$; $-1 + s = 9 + 3t$

First two give $s = 3$, $t = -2$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \dfrac{|-6|}{\sqrt{19}\sqrt{18}} = 0.3244$, $\varphi = 71.1°$

Answer: Lines do not meet (skew); angle 71.1°

10 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 5 kg lies on a smooth horizontal table, with a block of mass 3 kg on it ($\mu = 0.3$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when $P = 47.04$. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \dfrac{P}{8}$. The block is accelerated only by friction: $F = 3a \le \mu mg$

So $a \le \mu g = 2.94$, and P_max $= 8 \times 2.94 = 23.52$

$P = 47.04 > 23.52$, so the block slips. Block: $a = \mu g = 2.94$

Plank: $47.04 - 8.82 = 5a$, so $a = 7.644$

Answer: (a) 23.52 N (b) block 2.94 m s^{-2}, plank 7.644 m s^{-2}