Day 181 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that

$\frac{\sin 6\theta \tan 5\theta}{1 - \cos 2\theta}$

approaches as $\theta \to 0$.

$\sin 6\theta \approx 6\theta$, $\tan 5\theta \approx 5\theta$, $1 - \cos 2\theta \approx 2\theta^2$

Expression $\approx \frac{6\theta \times 5\theta}{2\theta^2} = 15$

Answer: 15

2 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 15 pairs of values: $\Sigma x = 210$, $\Sigma y = 720$, $\Sigma x^2 = 3072$, $\Sigma xy = 10476$.

(a) Find the equation of the regression line of y on x.

(b) The x values lie between 10 and 19. Estimate y when x = 22 and comment on its reliability.

$S_{xx} = 3072 - \frac{210^2}{15} = 132$

$S_{xy} = 10476 - \frac{210 \times 720}{15} = 396$

$b = \frac{S_{xy}}{S_{xx}} = \frac{396}{132} = 3$

$a = \bar{y} - b\,\bar{x} = 48 - (3)(14) = 6$

$y = 6 + 3x$

At x = 22: y = 72. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) y = 6 + 3x (b) 72, unreliable (extrapolation)

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 5 strips to estimate

$\int_1^2 \sqrt{x}e^x \, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{5}$; ordinates at $x = 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \ldots, 2$

$\text{Sum} = \frac{h}{2}[y_0 + y_5 + 2(y_1 + \ldots + y_4)] = 5.880$

f''(x) > 0 on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 5.880; overestimate

4 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 34.5 and variance 8.625.

(a) Find n and p.

(b) Find P(X = 32), correct to 3 significant figures.

$np = 34.5$ and $np(1 - p) = 8.625$

Divide: $1 - p = 8.625 \div 34.5 = \frac{1}{4}$, so $p = \frac{3}{4}$

$n = 34.5 \div \frac{3}{4} = 46$

$P(X = 32) = C(46, 32) \left(\frac{3}{4}\right)^{32} \left(\frac{1}{4}\right)^{14} = 0.0898$

Answer: (a) n = 46, p = $\frac{3}{4}$ (b) 0.0898

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5 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 4 cm, subtends an angle of $\frac{\pi}{3}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 4^2 \times \frac{\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 4\sqrt{3}$

Area = $\frac{8}{3}\pi - 4\sqrt{3}$

Chord AB = $2r\sin(\theta/2) = 4$

Arc AB = $r\theta = \frac{4}{3}\pi$

Perimeter = arc + chord

Answer: Area = $\frac{8}{3}\pi - 4\sqrt{3}$ cm^2, perimeter = $4 + \frac{4}{3}\pi$ cm

6 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-4 - 3\sqrt{6}}{4 + \sqrt{6}}$ in the form $p + q\sqrt{6}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 - \sqrt{6})$.

Numerator: $(-4 - 3\sqrt{6})(4 - \sqrt{6}) = 2 - 8\sqrt{6}$.

Denominator: $4^2 - 1^2(6) = 10$.

So the fraction = $\frac{2 - 8\sqrt{6}}{10}$.

Answer: $\frac{1}{5} - \frac{4}{5}\sqrt{6}$

7 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(-5\mathbf{i} - 2\mathbf{j})$ m with velocity $(6\mathbf{i} + 4\mathbf{j})$ m s^{-1}. At t = 6, P has velocity $(4\mathbf{j})$ m s^{-1}.

Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{i} + \mathbf{j})$, (c) the position vector of P at that time.

$\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t} = \frac{(4\mathbf{j}) - (6\mathbf{i} + 4\mathbf{j})}{6} = (-\mathbf{i})$

$\mathbf{v} = (6\mathbf{i} + 4\mathbf{j}) + t(-\mathbf{i})$

Parallel to $(\mathbf{i} + \mathbf{j})$: $(-t + 6) \times 1 = (4) \times 1$, giving t = 2

$\mathbf{r} = (-5\mathbf{i} - 2\mathbf{j}) + 2(6\mathbf{i} + 4\mathbf{j}) + \frac{1}{2} \times 2^2 (-\mathbf{i}) = (5\mathbf{i} + 6\mathbf{j})$

Answer: (a) $(-\mathbf{i})$ m s^{-2} (b) t = 2 (c) $(5\mathbf{i} + 6\mathbf{j})$ m

8 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_4 3$ is irrational.

Assume $\log_4 3 = \frac{m}{n}$, m, n positive integers.

Then $4^{\frac{m}{n}} = 3$, so $4^m = 3^n$.

Prime factors of 4^m and 3^n must be identical,

but 3 divides 3^n but 3 does not divide 4^m.

This is a contradiction, so $\log_4 3$ is irrational.

Answer: Proved

9 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 175 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{7}{24}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{3}{4}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 175\cos\alpha$, so $R = 175 \times \frac{24}{25} - P\sin\beta$

Along the plane: $P\cos\beta = 175\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 175(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{4}{5} + \frac{3}{4} \times \frac{3}{5}) = 175(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25})$, so $P = 140$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_{min} = 175(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25}) \div \frac{5}{4}$

Answer: (a) 140 N (b) 140 N

10 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 2 and its common difference d is non-zero. The 1st, 2nd and 7th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: 2, 2 + d, 2 + 6d.

Geometric: $(2 + d)^2 = (2)(2 + 6d)$.

Expand: $d^2 + 4d + 4 = 12d + 4$, so $d(d - 8) = 0$.

$d \neq 0$, so $d = 8$. Ratio $= \frac{10}{2} = 5$.

Answer: d = 8, ratio = 5