**Day 18** **A Level Maths: S Tier 10-a-day: Model Solutions**

## 1 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_3 x - \log_3 y = 2$
$x + y = 70$

$\log_3(x/y) = 2$, so $x = 3^2y = 9y$
$9y + y = 70$, so $10y = 70$
**Answer: x = 63, y = 7**

## 2 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 + n + 41$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.
For n = 40: $40^2 + 40 + 41 = 1681$.
$1681 = 41 \times 41$, which is not prime.
All smaller values of n give a prime, so n = 40 is the smallest.
**Answer: n = 40 (value 1681 = 41 × 41)**

## 3 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that
$\frac{1 - \cos 3\theta}{\theta \tan 3\theta}$
approaches as $\theta \to 0$.

$1 - \cos 3\theta \approx \frac{1}{2}(3\theta)^2 = \frac{9}{2}\theta^2$, $\tan 3\theta \approx 3\theta$

Expression $\approx \frac{\frac{9}{2}\theta^2}{\theta \times 3\theta} = \frac{3}{2}$

**Answer: $\frac{3}{2}$**

## 4 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(17\mathbf{i} + 34\mathbf{j})$ m s$^{-1}$ and $(11\mathbf{i} + 26\mathbf{j})$ m s$^{-1}$ respectively (i horizontal, j vertically upwards).
Both are still in flight after 3 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $17t\mathbf{i} + (34t - 4.9t^2)\mathbf{j}$; position of B: $11t\mathbf{i} + (26t - 4.9t^2)\mathbf{j}$
Subtracting, the $4.9t^2$ terms cancel: $AB = t(6\mathbf{i} + 8\mathbf{j})$
After 3 s: $AB = (18\mathbf{i} + 24\mathbf{j})$
Distance $= \sqrt{(18^2 + 24^2)} = 30$
The direction of AB is that of $(6\mathbf{i} + 8\mathbf{j})$, which does not depend on t: tan angle $= \frac{8}{6}$
**Answer: (a) 30 m (b) 53.1°**

## 5 Stratified sample then probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school has 240 students in Year 10, 160 in Year 11 and 80 in the sixth form.
A stratified sample of 24 students is taken.
(a) Find the number sampled from each group.
(b) Two students are chosen at random from the sample. Find the probability they are from the same group.

Sample sizes: $24 \times \frac{240}{480}$, $24 \times \frac{160}{480}$, $24 \times \frac{80}{480} = 12, 8, 4$
P(same) = 12 × 11 + 8 × 7 + 4 × 3/24 × 23
$= \frac{200}{552} = 0.362$
**Answer: (a) 12, 8, 4 (b) 0.362**

## 6 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = e^x$ and $y = 4e^{-x}$ meet at the point P.
Find the exact area of the region bounded by the two curves and the y-axis.

$e^x = 4e^{-x}$ gives $e^{2x} = 4$, so $e^x = 2$, $x = \ln 2$

Area $= \int_0^{\ln 2} (4e^{-x} - e^x)\,dx = [-4e^{-x} - e^x]$

$= (-2 - 2) - (-4 - 1)$

**Answer: 1**

## 7 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(9 + x)^{\frac{1}{2}}$ up to and including the term in $x^2$.
(b) Use $x = 0.04$ to estimate $\sqrt{9.04}$ to 5 decimal places.

(a) $(9 + x)^{\frac{1}{2}} = 3(1 + \frac{1}{9}x)^{\frac{1}{2}}$.

$= 3 + \frac{1}{6}x - \frac{1}{216}x^2 + \ldots$

(b) $9 + x$ with $x = 0.04$ gives 9.04.

Substitute: $3 + \frac{1}{6}(0.04) + (-\frac{1}{216})(0.04)^2$.

$= 3.00666$ (5 d.p.).

**Answer: (a) $3 + \frac{1}{6}x - \frac{1}{216}x^2$ (b) 3.00666**

## 8 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 3 kg moves under a single force F N, where $\mathbf{F} = (12t - 3)\mathbf{i} + (18t^2 - 15)\mathbf{j}$ and t is the time in seconds.
At t = 0, P is at the origin with velocity $(\mathbf{i} - 3\mathbf{j})$ m s$^{-1}$.
Find (a) the velocity of P at t = 3, (b) the position vector of P at t = 3.

$\mathbf{a} = \frac{\mathbf{F}}{3} = (4t - 1)\mathbf{i} + (6t^2 - 5)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\,dt = (2t^2 - t + 1)\mathbf{i} + (2t^3 - 5t - 3)\mathbf{j}$ (using $\mathbf{v}(0) = (\mathbf{i} - 3\mathbf{j})$)
At t = 3: $\mathbf{v} = (16\mathbf{i} + 36\mathbf{j})$

$\mathbf{r} = \int \mathbf{v}\,dt = (\frac{2}{3}t^3 - 0.5t^2 + t)\mathbf{i} + (0.5t^4 - 2.5t^2 - 3t)\mathbf{j}$

At t = 3: $\mathbf{r} = (16.5\mathbf{i} + 9\mathbf{j})$

**Answer: (a) $(16\mathbf{i} + 36\mathbf{j})$ m s$^{-1}$ (b) $(16.5\mathbf{i} + 9\mathbf{j})$ m**

## 9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 64)$. $H_0$: $\mu = 43$ is tested against $H_1$: $\mu > 43$ using the mean of a sample of size 25 at the 5% level.
(a) Find the critical region for the sample mean.
(b) Find the probability of a Type II error when $\mu = 46$.

Under $H_0$, sample mean $\sim N(43, \frac{64}{25})$, standard error = 1.600

$z = 1.6449$ for the upper tail: critical value $= 43 + 1.6449 \times 1.600 = 45.63$
Type II error: do not reject $H_0$, sample mean < 45.63, where mean $\sim N(46, 1.600)$

$z = \frac{45.63 - 46}{1.600} = -0.230$

Probability = 0.409

**Answer: (a) sample mean > 45.6 (b) 0.409**

## 10 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line $l_1$ passes through A and B.
Line $l_2$ is perpendicular to $l_1$ at B and meets the
x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{3-2}{-3} = -\frac{1}{3}$.

Gradient of $l_2 = 3$. Equation: $y - 3 = 3(x + 3)$.

$y = 0$: $x = -4$, so C$(-4, 0)$.

AB $= \sqrt{10}$, BC $= \sqrt{10}$: area $= \frac{1}{2} \times$ AB $\times$ BC.

**Answer: C$(-4, 0)$; area 5**