Day 179 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 5 m s^{-1} from the top of a cliff. The top of the cliff is 97.5 m above the sea. Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-97.5 = 5t - 4.9t^2$, so $4.9t^2 - 5t - 97.5 = 0$

$t = \frac{5 + \sqrt{(5^2 + 19.6 \times 97.5)}}{9.8} = \frac{5 + 44}{9.8} = 5$

Height risen above the cliff top = $\frac{5^2}{(2 \times 9.8)} = 1.28$ m

Total distance = 2 × 1.28 + 97.5

Answer: (a) 5 s (b) 100 m

2 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 10) + \log_2(x - 5) = 4$,
justifying clearly any root that you reject.

$(x + 10)(x - 5) = 2^4 = 16$

$x^2 + 5x - 66 = 0$

$(x - 6)(x + 11) = 0$, so $x = 6$ or $x = -11$

$x = -11$ makes $x + 10 = -1 < 0$ and log of a negative number is undefined

Answer: x = 6

3 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 5x^2 + 12x - 8$.

Show that $(x - 1)$ is a factor of f(x), and that the equation f(x) = 0 has exactly one real root.

$f(1) = 1 - 5 + 12 - 8 = 0$, so $(x - 1)$ is a factor.

$f(x) = (x - 1)(x^2 - 4x + 8)$.

For $x^2 - 4x + 8 = 0$: $b^2 - 4ac = 16 - 32 = -16 < 0$.

No real roots from the quadratic, so the only real root is x = 1.

Answer: Proved; the only real root is x = 1

4 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 68.6 m s^{-1}.
Find the length of time for which the ball is more than 220.5 m above the ground, and the greatest height it reaches. Take g = 9.8 m s^{-2}.

At height 220.5 m: $v^2 = 68.6^2 - 2 \times 9.8 \times 220.5 = 384.16$, so v = 19.6

Time from this height to the top: $t = \frac{19.6}{9.8} = 2$ s

By symmetry the time above this height is 2 × 2 = 4 s

Greatest height = $\frac{68.6^2}{(2 \times 9.8)} = 240.1$

Answer: 4 s, 240.1 m

5 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 27 and upper quartile 38. An outlier is a value more than 2 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 10 observations have mean 31. The largest is 63, an outlier, and it is removed. Find the new mean.

IQR = 38 − 27 = 11, so 2 × IQR = 22

Lower limit = 27 − 22 = 5, upper limit = 38 + 22 = 60

$\Sigma x = 10 \times 31 = 310$, so without 63: $\Sigma x = 247$

New mean = $\frac{247}{9}$

Answer: (a) 5 to 60 (b) 27.44

6 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $12y = 5x + k$ is a tangent to the circle $(x + 1)^2 + (y + 1)^2 = 4$.
Find the two possible values of k.

Substitute $y = \frac{5x + k}{12}$: $(x + 1)^2 + (\frac{5x + k}{12} + 1)^2 = 4$.

Multiply by 144 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 7)^2 = 676$.

$k + 7 = \pm 26$.

Answer: $k = -33$ or $k = 19$

7 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} -7 \\ -9 \\ -1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -7 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $-7 + 2s = -7 - 3t$; $-9 + 3s = 1$; $-1 + s = 4 + t$

First two give $s = 10/3$, $t = -20/9$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \frac{|-5|}{\sqrt{14}\sqrt{10}} = 0.4226$, $\varphi = 65.0°$

Answer: Lines do not meet (skew); angle 65.0°

8 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 2x^2 - 4x - 4$ for $x \le k$. Given that k is the greatest value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 2(x - 1)^2 - 6$.

The vertex is at x = 1, so k = 1 (f is one-one on one side of it).

Let $y = f(x)$: $y + 6 = 2(x - 1)^2$, so $(x - 1) = \pm\sqrt{\frac{y + 6}{2}}$.

As $x \le 1$, take the negative root: $x = 1 - \sqrt{\frac{y + 6}{2}}$.

Range of f = domain of f^{-1}: $f(1) = -6$, so $x \ge -6$.

Answer: $k = 1$; $f^{-1}(x) = 1 - \sqrt{\frac{x + 6}{2}}$; $x \ge -6$

9 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 12-sided dice, numbered 1 to 12, are rolled. M is the larger of the two scores.

(a) Show that $P(M = m) = \frac{(2m - 1)}{144}$ for m = 1, 2, ..., 12.

(b) Find P(M = 9) and E(M).

(a) $M \le m$ means both scores $\le m$: $P(M \le m) = (m/12)^2$, so $P(M = m) = \frac{m^2 - (m - 1)^2}{144} = \frac{(2m - 1)}{144}$

(b) $P(M = 9) = \frac{17}{144}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{144}\Sigma\, m(2m - 1)$

$= \frac{611}{72}$

Answer: (b) $P(M = 9) = \frac{17}{144}$, $E(M) = \frac{611}{72}$

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10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = -2xy$, and y = 5 when x = 0.

Solve the differential equation and find y when x = 2.

$\int \frac{1}{y}\, dy = \int -2x\, dx$, so $\ln y = -2x^2/2 + c = -1x^2 + c$

x = 0, y = 5: c = ln 5, so $y = 5\, e^{-1x^2}$

x = 2: exponent = −4

Answer: $y = 5\, e^{-1x^2}$; $y = 5e^{-4}$