Day 177 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(3\mathbf{i} - 5\mathbf{j})$ m with velocity $(-3\mathbf{i})$ m s^{-1}. At t = 10, P has velocity $(17\mathbf{i} + 10\mathbf{j})$ m s^{-1}.
Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{i} + 2\mathbf{j})$, (c) the position vector of P at that time.

$a = \frac{v - u}{t} = \frac{(17\mathbf{i} + 10\mathbf{j}) - (-3\mathbf{i})}{10} = (2\mathbf{i} + \mathbf{j})$

$v = (-3\mathbf{i}) + t(2\mathbf{i} + \mathbf{j})$

Parallel to $(\mathbf{i} + 2\mathbf{j})$: $(2t - 3) \times 2 = (t) \times 1$, giving t = 2

$r = (3\mathbf{i} - 5\mathbf{j}) + 2(-3\mathbf{i}) + \frac{1}{2} \times 2^2 (2\mathbf{i} + \mathbf{j}) = (\mathbf{i} - 3\mathbf{j})$

Answer: (a) $(2\mathbf{i} + \mathbf{j})$ m s^{-2} (b) t = 2 (c) $(\mathbf{i} - 3\mathbf{j})$ m

2 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact gradient of the curve $y = 3^{3x + 2}$
at the point where x = 0.

$y = e^{(3x + 2)\ln 3}$

$dy/dx = 3 \ln 3 \times 3^{3x + 2}$

At x = 0: exponent = 2, so $dy/dx = 3 \ln 3 \times 3^2$

Answer: $27\ln 3$

3 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 = 0$.

Let $u = x^{\frac{1}{3}}$: $u^2 + u - 2 = 0$.
$(u + 2)(u - 1) = 0$, so $u = -2$ or $u = 1$.
Cube: $x = -8$ or $x = 1$.
Answer: $x = -8$ or $x = 1$

4 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{4}{5}$. A force P N acts up a line of greatest slope; the particle stays at rest for $32.5 \leq P \leq 47.5$.
Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 95. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 47.5 and up the plane at P = 32.5
$P_{max} = W \sin\alpha + F$ and $P_{min} = W \sin\alpha - F$, where $F = \mu W \cos\alpha$

Adding: $2W \sin\alpha = 80$, so $W = \frac{80}{2 \times \frac{4}{5}} = 50$

Subtracting: $2F = 15$, so $F = 7.5 = \mu \times 50 \times \frac{3}{5}$, giving $\mu = 0.25$

P = 95: net force up the plane = 95 − 47.5 = 47.5, and mass = $\frac{50}{9.8}$

Answer: (a) 50 N (b) 0.25 (c) 9.31 m s^{-2}

5 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $12y = 5x + k$ is a tangent to the circle
$(x + 3)^2 + (y + 1)^2 = 1$.
Find the two possible values of k.

Substitute $y = \frac{5x + k}{12}$: $(x + 3)^2 + (\frac{5x + k}{12} + 1)^2 = 1$.

Multiply by 144 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k - 3)^2 = 169$.

$k - 3 = \pm 13$.

Answer: $k = -10$ or $k = 16$

6 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 15 pairs of values: $\Sigma x = 150$, $\Sigma y = 750$, $\Sigma x^2 = 1649$, $\Sigma xy = 7798$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 6 and 15. Estimate y when x = 14 and comment on its reliability.

$S_{xx} = 1649 - \frac{150^2}{15} = 149$

$S_{xy} = 7798 - \frac{150 \times 750}{15} = 298$

$b = \frac{S_{xy}}{S_{xx}} = \frac{298}{149} = 2$

$a = \bar{y} - b\,\bar{x} = 50 - (2)(10) = 30$

$y = 30 + 2x$

At x = 14: y = 58. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) $y = 30 + 2x$ (b) 58, unreliable (extrapolation)

7 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x e^y + 3y = 1$.
Find the equation of the normal to C at the point (1, 0),
in the form ax + by + c = 0 with integers a, b, c.

Differentiate implicitly with respect to x

At (1, 0): $dy/dx = -\frac{1}{4}$

Normal gradient $= -1 \div (-\frac{1}{4}) = 4$

$y - 0 = 4(x - 1)$, then clear fractions

Answer: $4x - y - 4 = 0$

8 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased spinner lands on 'win' with the same unknown probability p each time. It is used 7 times in round 1 and 6 times in round 2.
Given that exactly 9 wins occur altogether, find the probability that exactly 6 of them were in round 1.

Let $X \sim B(7, p)$, $Y \sim B(6, p)$ be independent; $X + Y \sim B(13, p)$

$P(X = 6 \mid X + Y = 9) = \frac{P(X = 6)P(Y = 3)}{P(X + Y = 9)}$

$= \frac{C(7,6)C(6,3)\, p^9(1-p)^4}{C(13,9)\, p^9(1-p)^4}$

The powers of p and $(1 - p)$ cancel, leaving $\frac{7 \times 20}{715}$

Answer: $\frac{28}{143} = 0.196$

9 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{33 + 20\sin\theta + 21\cos\theta}$

(a) Write $20\sin\theta + 21\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{20^2 + 21^2} = 29$, $\tan\alpha = \frac{21}{20}$, $\alpha = 46.4°$

Denominator $= 33 + 29\sin(\theta + 46.4°)$, which lies between 4 and 62

Greatest $f(\theta) = \frac{4}{4} = 1$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{62} = \frac{2}{31}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 46.4° = 270°$, so $\theta = 223.6°$

Answer: (a) $29\sin(\theta + 46.4°)$ (b) greatest 1, least $\frac{2}{31}$, $\theta = 223.6°$

10 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 4x^2 - 16x + 23$ for $x \geq k$. Given that k is the least value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 4(x - 2)^2 + 7$.
The vertex is at $x = 2$, so $k = 2$ (f is one-one on one side of it).

Let $y = f(x)$: $y - 7 = 4(x - 2)^2$, so $(x - 2) = \pm\sqrt{\frac{y - 7}{4}}$.

As $x \geq 2$, take the positive root: $x = 2 + \sqrt{\frac{y - 7}{4}}$.

Range of f = domain of f^{-1}: $f(2) = 7$, so $x \geq 7$.

Answer: $k = 2$; $f^{-1}(x) = 2 + \sqrt{\frac{x - 7}{4}}$; $x \geq 7$