Day 173 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 3x^2 + x - 3$.

Show that $(x - 3)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(3) = 27 - 27 + 3 - 3 = 0$, so $(x - 3)$ is a factor.

$f(x) = (x - 3)(x^2 + 1)$.

For $x^2 + 1 = 0$: $b^2 - 4ac = 0 - 4 = -4 < 0$.

No real roots from the quadratic, so the only real root is $x = 3$.

Answer: Proved; the only real root is $x = 3$

2 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 40 m from O, and reaches a maximum height of 15 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take $g = 9.8$ m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 15$, so $u_y = 17.15$

Time of flight $= \frac{2u_y}{9.8} = 3.499$ s, so $u_x = \frac{40}{3.499} = 11.43$

$\tan\theta = \frac{u_y}{u_x} = 1.500$, so $\theta = 56.3°$

Speed $= \sqrt{(u_x^2 + u_y^2)} = 20.6$

Answer: $\theta = 56.3°$, speed 20.6 m s^{-1}

3 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 14 and variance 8.4.

(a) Find n and p.

(b) Find P(X = 15), correct to 3 significant figures.

$np = 14$ and $np(1 - p) = 8.4$

Divide: $1 - p = 8.4 \div 14 = \frac{3}{5}$, so $p = \frac{2}{5}$

$n = 14 \div \frac{2}{5} = 35$

$P(X = 15) = C(35, 15) (\frac{2}{5})^{15} (\frac{3}{5})^{20} = 0.128$

Answer: (a) $n = 35$, $p = \frac{2}{5}$ (b) 0.128

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $12\sec^2\theta + 22\tan\theta - 6 = 0$

for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $12\tan^2\theta + 22\tan\theta + 6 = 0$

$2(2\tan\theta + 3)(3\tan\theta + 1) = 0$

$\tan\theta = -\frac{3}{2}$ or $\tan\theta = -\frac{1}{3}$

$\theta = 123.7°$ and $161.6°$ $(+180°)$

Answer: $\theta = 123.7°, 161.6°, 303.7°, 341.6°$

5 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\sqrt{11}$ is irrational.

Assume $\sqrt{11} = \frac{a}{b}$ with a, b integers and no common factor.

Then $11 b^2 = a^2$, so a^2 is a multiple of 11.

As 11 is prime, a is a multiple of 11: let $a = 11k$.

Then $11 b^2 = 121 k^2$, so $b^2 = 11 k^2$, and b is a multiple of 11.

a and b have common factor 11: a contradiction. Hence $\sqrt{11}$ is irrational.

Answer: Proved

6 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 0.75 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 0.75 m s^{-2} to rest.
The journey takes 60 s and covers 432 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{0.75}$, time decelerating = $\frac{V}{0.75}$, so time at constant speed = $60 - \frac{V}{0.75} - \frac{V}{0.75}$

Area under the speed-time graph: $432 = 60V - \frac{V^2}{1.5} - \frac{V^2}{1.5}$

Multiply by 2ab: $1.5V^2 - 67.5V + 486 = 0$
V = 9 or V = 36
V = 36 gives time at constant speed −36 s, which is negative, so V = 9
Time at constant speed = 60 − 12 − 12 = 36 s
Answer: V = 9, 36 s

7 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(9, $\frac{2}{3}$).

(a) Show that P(X = r + 1) ÷ P(X = r) = $\frac{9 - r}{r + 1} \times 2$.
(b) Find the most likely value of X and its probability (3 s.f.).

P(X = r + 1) ÷ P(X = r) = $\frac{C(9, r + 1)}{C(9, r)} \times \frac{p}{q} = \frac{9 - r}{r + 1} \times \frac{p}{q}$

With p = $\frac{2}{3}$ and q = $\frac{1}{3}$, p ÷ q = 2

The ratio exceeds 1 when $(9 - r)\frac{2}{3} > (r + 1)\frac{1}{3}$, that is $r < \frac{17}{3}$

So probabilities rise up to r = 6 and then fall; the mode is 6
P(X = 6) = 0.273
Answer: 6 with probability 0.273

8 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{2x} \sin 2x$. Show that $dy/dx = e^{2x}(2 \sin 2x + 2 \cos 2x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (2)e^{2x} \sin 2x + e^{2x} \times 2 \cos 2x$
dy/dx = 0 requires $2 \sin 2x + 2 \cos 2x = 0$
tan 2x = −1, smallest positive 2x = 2.3562
x = 1.18
Second derivative test (or gradient change): a maximum
Answer: x = 1.18, maximum

9 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{22 + 8\sqrt{6}}$ in the form
$a + b\sqrt{6}$, where a and b are positive integers.

Suppose $\sqrt{22 + 8\sqrt{6}} = a + b\sqrt{6}$. Square both sides.
$22 + 8\sqrt{6} = a^2 + 6b^2 + 2ab\sqrt{6}$.
Equate: $a^2 + 6b^2 = 22$ and 2ab = 8, so ab = 4.
Try a = 4, b = 1: 16 + 6 = 22 □.
(a > $b\sqrt{6}$, so the root is positive.)
Answer: $4 + 1\sqrt{6}$

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10 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^2 xe^{3x}\,dx$

Let u = x and dv/dx = e^{3x}

Integral = [uv] $-\int$ v du/dx dx

Evaluate between the limits

Answer: $\frac{1}{9} + \frac{5}{9}e^6$