Day 172 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-t^2 + 7t)\mathbf{i} + (-1.5t^2 + 8t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(\mathbf{i} + \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2t + 7)\mathbf{i} + (-3t + 8)\mathbf{j}$

Parallel to $(\mathbf{i} + \mathbf{j})$: $(-2t + 7) \times 1 = (-3t + 8) \times 1$

$t(1) = 1$, so $t = 1$

At $t = 1$: $\mathbf{v} = (5\mathbf{i} + 5\mathbf{j})$, speed $= \sqrt{(5^2 + 5^2)}$

Answer: (a) $t = 1$ (b) 7.07 m s^{-1}

2 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 5 strips to estimate

$\int_1^2 \sqrt{4 - x^2}\, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{5}$; ordinates at $x = 1, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \ldots, 2$

Sum $= \frac{h}{2}[y_0 + y_5 + 2(y_1 + \ldots + y_4)] = 1.193$

$f''(x) < 0$ on the interval, so the curve is concave and the trapezia lie below the curve

Answer: 1.193; underestimate

3 PMCC from summary data and a hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

For 8 pairs of values, $S_{xx} = 144$, $S_{yy} = 441$ and $S_{xy} = 160$.

(a) Calculate the product moment correlation coefficient r.

(b) The critical value for a two-tailed test at the 5% level with n = 8 is 0.7067. Carry out the test of H_0: $\rho = 0$ against H_1: $\rho \neq 0$.

$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{160}{\sqrt{(144 \times 441)}} = \frac{160}{252} = 0.6349$

H_0: $\rho = 0$, H_1: $\rho \neq 0$

$|r| = 0.6349 <$ the critical value 0.7067

Do not reject H_0: there is insufficient evidence of correlation between the variables

Answer: (a) r = 0.635 (b) do not reject H_0

4 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $5\cos\theta + 12\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.

(b) Hence solve $5\cos\theta + 12\sin\theta = 2$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{5^2 + 12^2} = 13$ and $\tan\alpha = \frac{12}{5}$, so $\alpha = 67.4°$

So $5\cos\theta + 12\sin\theta \equiv 13\cos(\theta - 67.4°)$

$\cos(\theta - \alpha) = \frac{2}{13}$

$\theta - \alpha = \pm 81.2°$ (using $\alpha = 67.4°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 13, α = 67.4° (b) θ = 148.5°, 346.2°

5 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both

$x^2 + 4x - 5 \leq 0$ and $-x - 1 > 3$.

Quadratic: $(x + 5)(x - 1) \leq 0$ gives $-5 \leq x \leq 1$.

Linear: $-x - 1 > 3$ gives $x < -4$. (the inequality reverses as we divide by a negative)

Both must hold: take the overlap of the two sets.

Answer: $-5 \leq x < -4$

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6 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, P(A) = 0.5, P(B) = 0.6 and P(A | B) = 0.4.
(a) Find P(A ∩ B).
(b) Find P(A′ | B′).

P(A ∩ B) = P(A | B) × P(B) = 0.4 × 0.6 = 0.24
P(A ∪ B) = 0.5 + 0.6 − 0.24 = 0.86
P(B′) = 0.4
P(A′ ∩ B′) = 1 − P(A ∪ B) = 0.14
$P(A' \mid B') = \frac{0.14}{0.4}$

Answer: (a) 0.24 (b) $\frac{7}{20}$

7 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - 2y = -3$
$x^2 - 2xy + y^2 = 16$

From the first equation $x = -3 + 2y$.
Substitute into the second equation and collect terms:
this gives $y^2 - 6y - 7 = 0$.
$y = -1$ or $y = 7$, giving $x = -5$ or $x = 11$.
Answer: (x, y) = (−5, −1) and (11, 7)

8 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 180 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{12}{5}$, $\mu = \frac{5}{12}$) by a force P N at angle β above the plane.
(a) Find P when $\tan\beta = \frac{5}{12}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 180\cos\alpha$, so $R = 180 \times \frac{5}{13} - P\sin\beta$
Along the plane: $P\cos\beta = 180\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 180(\sin\alpha + \mu\cos\alpha)$
(a) $P(\frac{12}{13} + \frac{5}{12} \times \frac{5}{13}) = 180(\frac{12}{13} + \frac{5}{12} \times \frac{5}{13})$, so P = 180
(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1+\mu^2)} = \frac{13}{12}$ (when $\tan\beta = \frac{5}{12}$)
$P_{min} = 180(\frac{12}{13} + \frac{5}{12} \times \frac{5}{13}) \div \frac{13}{12}$
Answer: (a) 180 N (b) 180 N

9 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{9}$ and $\tan B = \frac{4}{5}$.

Find tan(A + B) and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{9} + \frac{4}{5}}{1 - \frac{4}{45}} = \frac{\frac{41}{45}}{\frac{41}{45}} = 1$$

A + B is acute, so $A + B = \pi/4$
Answer: tan(A + B) = 1, A + B = $\frac{\pi}{4}$

10 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = -\frac{1}{3}u_n + 8$, $u_1 = -14$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.01$.

(a) At the limit $L = -\frac{1}{3}L + 8$, so $L(1 - (-\frac{1}{3})) = 8$.

$L = 6$.

(b) $u_n - L = (u_1 - L)((-\frac{1}{3}))^{n-1} = (-20)((-\frac{1}{3}))^{n-1}$.

Need $20 \times (\frac{1}{3})^{n-1} < 0.01$, so $n - 1 > 6.919$.

The least n is 8.

Answer: (a) L = 6 (b) n = 8