Day 171 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x + y = 2$

$x^2 + 2xy + 2y^2 = 13$

From the first equation $x = 2 - y$.

Substitute into the second equation and collect terms:

this gives $y^2 - 9 = 0$.

$y = -3$ or $y = 3$, giving $x = 5$ or $x = -1$.

Answer: (x, y) = (5, −3) and (−1, 3)

2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A radioactive sample has mass $m = m_0e^{-kt}$ grams, t in days.

Initially $m = 160$ and after 5 days $m = \frac{320}{3}$.

Find k, and the time for the mass to fall to 40 grams, to 3 s.f.

$m_0 = 160$. $160\ e^{-5k} = \frac{320}{3}$ so $e^{-5k} = \frac{2}{3}$

$k = \frac{1}{5} \ln \frac{3}{2} = 0.0811$

$160\ e^{-kt} = 40$ so $e^{kt} = 4$, $t = \frac{1}{k} \ln 4$

$t = 17.1$ days

Answer: k = 0.0811, t = 17.1 days

3 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{4}{5}u_n + 2$, $u_1 = -14$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = \frac{4}{5}L + 2$, so $L(1 - \frac{4}{5}) = 2$.

$L = 10$.

(b) $u_n - L = (u_1 - L)(\frac{4}{5})^{n-1} = (-24)(\frac{4}{5})^{n-1}$.

Need $24 \times (\frac{4}{5})^{n-1} < 0.001$, so $n - 1 > 45.199$.

The least n is 47.

Answer: (a) L = 10 (b) n = 47

4 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.

Find (a) the total number of runners, (b) an estimate of the number with $9 \leq t < 33$.

Frequency = frequency density × width. Bar 22 to 32: height 1 squares, width 10, frequency 10

So one square of frequency density = 1 runners per second

Class frequencies = 42, 12, 8, 10, 8

Total = 80

Estimate = 1 × [1(9) + 2(4) + 1(10) + 1(1)] = 28

Answer: (a) 80 (b) 28

5 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/3} \sin 3x \cos x \, dx$

Use $2 \sin A \cos B \equiv \sin(A + B) + \sin(A - B)$

Integrate term by term

Substitute the limits

Answer: $\frac{9}{16}$

6 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1500 kg tows a trailer of mass 500 kg along a horizontal road, with driving force 1800 N. Resistances are 600 N on the car and 400 N on the trailer.

(a) Find the tension in the tow bar. At 6.4 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1800 - 600 - 400}{2000} = 0.4$

Trailer: $T - 400 = 500 \times 0.4$, so $T = 600$

After the break the trailer decelerates at $\frac{400}{500} = 0.8$, and stops after $t = \frac{6.4}{0.8} = 8$ s

Trailer travels $\frac{6.4}{2} \times 8 = 25.6$ m

Car: $a = \frac{1800 - 600}{1500} = 0.8$, so $s = 6.4 \times 8 + \frac{1}{2} \times 0.8 \times 8^2 = 76.8$ m

Distance apart = $76.8 - 25.6$

Answer: (a) 600 N (b) 51.2 m

7 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $20 \sin \theta + 21 \cos \theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.

(b) Hence solve $20 \sin \theta + 21 \cos \theta = 7$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{20^2 + 21^2} = 29$ and $\tan \alpha = \frac{21}{20}$, so $\alpha = 46.4°$

So $20 \sin \theta + 21 \cos \theta \equiv 29 \sin(\theta + 46.4°)$

$\sin(\theta + \alpha) = \frac{7}{29}$

$\theta + \alpha = 14.0°$ or $166.0°$ (using $\alpha = 46.4°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) R = 29, α = 46.4° (b) θ = 119.6°, 327.6°

8 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 4 and its common difference d is non-zero. The 1st, 3rd and 7th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: $4, 4 + 2d, 4 + 6d$.

Geometric: $(4 + 2d)^2 = (4)(4 + 6d)$.

Expand: $4d^2 + 16d + 16 = 24d + 16$, so $d(4d - 8) = 0$.

$d \neq 0$, so $d = 2$. Ratio $= \frac{8}{4} = 2$.

Answer: d = 2, ratio = 2

9 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 10 pairs of values: $\Sigma x = 150$, $\Sigma y = 400$, $\Sigma x^2 = 2308$, $\Sigma xy = 5884$.

(a) Find the equation of the regression line of y on x.

(b) The x values lie between 11 and 20. Estimate y when x = 19 and comment on its reliability.

$S_{xx} = 2308 - \frac{150^2}{10} = 58$

$S_{xy} = 5884 - \frac{150 \times 400}{10} = -116$

$b = \frac{S_{xy}}{S_{xx}} = \frac{-116}{58} = -2$

$a = \bar{y} - b\,\bar{x} = 40 - (-2)(15) = 70$

$y = 70 - 2x$

At x = 19: y = 32. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) $y = 70 - 2x$ (b) 32, unreliable (extrapolation)

10 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 11 kg and 9 kg hang on a light string over a smooth fixed pulley. The 11 kg particle is 0.5 m above the ground; the 9 kg one is on the ground.

Released from rest, find (a) the tension, (b) the greatest height reached by the 9 kg particle. Take $g = 9.8$ m s^{-2}.

$11g - T = 11a$ and $T - 9g = 9a$, so $a = \frac{11 - 9}{11 + 9} \times 9.8 = 0.98$

$T = 11(9.8 - 0.98) = 97.02$

The 9 kg particle rises 0.5 m before the 11 kg particle lands, reaching $v^2 = 2 \times 0.98 \times 0.5 = 0.98$

The string goes slack, so it then moves freely under gravity: extra height $= \frac{v^2}{2g} = \frac{0.98}{19.6} = 0.05$

Greatest height = 0.5 + 0.05

Answer: (a) 97.02 N (b) 0.55 m