Day 170 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A manufacturer says that 50% of its screens have no flaws. A sample of 25 screens is checked, and 18 have no flaws. Test, at the 5% significance level, the claim that the proportion is higher.
Give your conclusion in context.

H_0: p = 0.5, H_1: p > 0.5. Let X ~ B(25, 0.5)
$P(X \geq 18) = 1 - P(X \leq 17) = 0.0216$
0.0216 is less than 5%, so reject H_0
There is sufficient evidence that the proportion is higher than the claimed 0.5
Answer: Reject H_0: p-value 0.0216 < 5%

2 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:
$8^x \times 2^y = 128$
$9^x \div 3^y = 27$

Write each side with prime bases: $128 = 2^7$, $27 = 3^3$.
First: $2^{3x} \times 2^{1y} = 2^7$, so 3x + 1y = 7.
Second: $3^{2x - 1y} = 3^3$, so 2x − 1y = 3.
Solving the pair of linear equations: x = 2, y = 1.
Answer: x = 2, y = 1

3 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
lg x + lg y = 2
x − y = 99

lg(xy) = 2, so $xy = 10^2 = 100$
x = y + 99: $y^2 + 99y - 100 = 0$
(y − 1)(y + 100) = 0; y > 0 so y = 1
Then x = y + 99
Answer: x = 100, y = 1

4 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-7x^2 + 29x + 6}{(x + 3)(x - 3)^2}$ in the form
$\frac{A}{(x + 3)} + \frac{B}{(x - 3)} + \frac{C}{(x - 3)^2}$.

$-7x^2 + 29x + 6 \equiv A(x - 3)^2 + B(x + 3)(x - 3) + C(x + 3)$.
Put x = −3: $-144 = A(-6)^2$, so A = −4.
Put x = 3: 30 = C(6), so C = 5.
Compare x^2: −7 = 1A + 1B, so B = −3.
Answer: A = −4, B = −3, C = 5

5 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 3 red counters and some black counters, n in total.
Two are taken without replacement. P(at least one red) = $\frac{36}{91}$. Find n.

P(neither red) = $\frac{(n - 3)}{n} \times \frac{(n - 4)}{(n - 1)} = \frac{55}{91}$
$\frac{55}{91} n(n - 1) = (n - 3)(n - 4)$
Rearranging gives a quadratic in n whose roots are n = 14 and one other value
The other root is $\frac{13}{6}$ so is rejected (not a valid number of counters)
n = 14
Answer: n = 14

6 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where $v = 6\sqrt{t} - 1t$.

Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{6}{2\sqrt{t}} - 1 = 0$ when $\sqrt{t} = 3$, so $t = 9$

$v = 6 \times 3 - 1 \times 9 = 9$

$v = 0$ when $\sqrt{t}(6 - 1\sqrt{t}) = 0$: $t = 36$ ($t = 0$ is the start)

$s = \int v\,dt = \frac{2}{3} \times 6\, t^{\frac{3}{2}} - \frac{1}{2}t^2$, so at $t = 36$: $s = 864 - 648$

Answer: (a) 9 m s^{-1} (b) 216 m

7 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{13}$ and $\tan B = \frac{6}{7}$.

Find $\tan(A + B)$ and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{13} + \frac{6}{7}}{1 - \frac{6}{91}} = \frac{\frac{85}{91}}{\frac{85}{91}} = 1$$

$A + B$ is acute, so $A + B = \pi/4$

Answer: $\tan(A + B) = 1$, $A + B = \frac{\pi}{4}$

8 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The volume of a sphere increases at a constant rate of 72π $cm^3 s^{-1}$.

When the radius is 6 cm, find the rate of increase of (a) the radius,

(b) the surface area. Volume = $\frac{4}{3}\pi r^3$.

$dV/dt = 4\pi r^2\, dr/dt$, so $72\pi = 4\pi \times 36 \times dr/dt$

$dr/dt = \frac{1}{2}$

$A = 4\pi r^2$, so $dA/dt = 8\pi r\, dr/dt = 8\pi \times 6 \times \frac{1}{2}$

Answer: (a) $\frac{1}{2}$ cm s^{-1} (b) 24π $cm^2 s^{-1}$

9 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|-x - 5| > 2x - 1$.

Boundary values: $-x - 5 = 2x - 1$ gives $x = -\frac{4}{3}$;

$-x - 5 = -(2x - 1)$ gives $x = 6$.

Also the corner of $|-x - 5|$ is at $x = -5$.

Testing each region (check the graph of each side) gives the solution set.

Answer: $x < 6$

10 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (4 m, 120 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{4}{3}$, where α is the angle between string and rod. A 40 N particle hangs at P, AP = 3 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Moments about A: $T\sin\alpha \times 4 = 120 \times 2 + 40 \times 3$

$T \times \frac{4}{5} \times 4 = 360$, so $T = 112.5$

Horizontal: $H = T\cos\alpha = 67.5$. Vertical: $V + T\sin\alpha = 120 + 40$, so $V = 70$

Force $= \sqrt{(67.5^2 + 70^2)} = 97.2$, angle $= \tan^{-1}(\frac{70}{67.5}) = 46.0°$ above the horizontal

Answer: (a) 112.5 N (b) 97.2 N at 46.0° above the horizontal (away from the wall)