Day 17 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 49 m s^{-1}.
Find the length of time for which the ball is more than 117.6 m above the ground, and the greatest height it reaches. Take g = 9.8 m s^{-2}.

At height 117.6 m: $v^2 = 49^2 - 2 \times 9.8 \times 117.6 = 96.04$, so v = 9.8

Time from this height to the top: $t = \frac{9.8}{9.8} = 1$ s

By symmetry the time above this height is 2 × 1 = 2 s

Greatest height = $\frac{49^2}{(2 \times 9.8)} = 122.5$

Answer: 2 s, 122.5 m

2 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{2x - 4} = x - 6$.

Square both sides: $2x - 4 = (x - 6)^2$.
$x^2 - 14x + 40 = 0$, so x = 4 or x = 10.
Check in $\sqrt{2x - 4} = x - 6$: x = 10 gives $4 \geq 0$ (valid).
x = 4 gives right-hand side $-2 < 0$, so it is rejected.
Answer: x = 10

3 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of
$y = 4x - 8 \sin x$ for $0 \leq x \leq 2\pi$,
and determine their nature.

$dy/dx = 4 - 8 \cos x = 0$, so $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$

$d^2y/dx^2 = 8 \sin x$
Its sign at each point decides maximum or minimum
Answer: $(\frac{\pi}{3}, \frac{4}{3}\pi - 4\sqrt{3})$ minimum; $(\frac{5\pi}{3}, \frac{20}{3}\pi + 4\sqrt{3})$ maximum

4 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{3 + 5\sqrt{2}}{2 + 3\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(2 - 3\sqrt{2})$.
Numerator: $(3 + 5\sqrt{2})(2 - 3\sqrt{2}) = -24 + \sqrt{2}$.
Denominator: $2^2 - 3^2(2) = -14$.
So the fraction = $\frac{-24 + \sqrt{2}}{-14}$.
Answer: $\frac{12}{7} - \frac{1}{14}\sqrt{2}$

5 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The size of a battery is $N(120, 5^2)$ hours. A battery is 'large' if it exceeds 127.
(a) Find the probability that a battery is large.
(b) 15 batterys are chosen at random. Find the probability that at least 4 are large.

(a) P(X > 127) = P(Z > 1.400) = 0.08076
(b) Let Y = number large, Y ~ B(15, 0.08076)
$P(Y \geq 4) = 1 - P(Y \leq 3) = 0.0282$
Answer: (a) 0.0808 (b) 0.0282

6 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where $v = 24\sqrt{t} - 6t$.
Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{24}{2\sqrt{t}} - 6 = 0$ when $\sqrt{t} = 2$, so $t = 4$

$v = 24 \times 2 - 6 \times 4 = 24$

$v = 0$ when $\sqrt{t}(24 - 6\sqrt{t}) = 0$: $t = 16$ ($t = 0$ is the start)

$s = \int v\,dt = \frac{2}{3} \times 24\, t^{\frac{3}{2}} - \frac{6}{2}t^2$, so at $t = 16$: $s = 1024 - 768$

Answer: (a) 24 m s^{-1} (b) 256 m

7 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{x^2 + 1}{x + 2}$ at the point where $x = 1$
meets the axes at A and B. Find the exact area of triangle OAB,
where O is the origin.

At $x = 1$: $y = \frac{2}{3}$. Differentiate: gradient $= \frac{4}{9}$

Tangent: $y - \frac{2}{3} = \frac{4}{9}(x - 1)$, so $y = \frac{4}{9}x + \frac{2}{9}$

y-intercept $\frac{2}{9}$; x-intercept $-\frac{1}{2}$

Area $= \frac{1}{2} \times |\frac{2}{9}| \times |-\frac{1}{2}|$

Answer: $\frac{1}{18}$

8 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A student's time on a task, in minutes, is $N(63, 8^2)$.
Given that it exceeds 55, find the probability that it exceeds 63.

Required $= P(X > 63 \mid X > 55) = \frac{P(X > 63)}{P(X > 55)}$

$P(X > 63) = P(Z > 0.000) = 0.5000$

$P(X > 55) = P(Z > -1.000) = 0.8413$

$= \frac{0.5000}{0.8413} = 0.594$

Answer: 0.594

9 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 2x^2 - 4x + 4$
for $x \geq k$. Given that k is the least value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 2(x - 1)^2 + 2$.
The vertex is at $x = 1$, so $k = 1$ (f is one-one on one side of it).

Let $y = f(x)$: $y - 2 = 2(x - 1)^2$, so $(x - 1) = \pm\sqrt{\frac{y - 2}{2}}$.

As $x \geq 1$, take the positive root: $x = 1 + \sqrt{\frac{y - 2}{2}}$.

Range of f = domain of f^{-1}: $f(1) = 2$, so $x \geq 2$.

Answer: $k = 1$; $f^{-1}(x) = 1 + \sqrt{\frac{x - 2}{2}}$; $x \geq 2$

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 11 \\ -7 \\ 9 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $11 - 2s = 3 - 2t$; $-7 + 2s = -2 - t$; $9 - 2s = 2 - t$

Solving the first two: $s = 3$, $t = -1$; the third equation is satisfied

Intersection: $(5, -1, 3)$

$\cos\varphi = \frac{|4|}{\sqrt{12}\sqrt{6}} = 0.4714$, $\varphi = 61.9°$

Answer: Lines meet at $(5, -1, 3)$; angle 61.9°