Day 168 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_2 7$ is irrational.

Assume $\log_2 7 = \frac{m}{n}$, m, n positive integers.

Then $2^{\frac{m}{n}} = 7$, so $2^m = 7^n$.

Prime factors of 2^m and 7^n must be identical,

but 7 divides 7^n but 7 does not divide 2^m.

This is a contradiction, so $\log_2 7$ is irrational.

Answer: Proved

2 Coded data: mean, standard deviation, Σx² (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 50 observations x, $\Sigma(x - 20) = 20$ and $\Sigma(x - 20)^2 = 908$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 20) = \frac{20}{50} = 0.4$, so mean of $x = 20.4$

Variance $= \frac{908}{50} - (0.4)^2 = 18$

$\sigma = \sqrt{18}$

$\Sigma x^2 = \Sigma(x - 20)^2 + 40\,\Sigma(x - 20) + 50 \times 20^2$

$= 908 + 800 + 20000$

Answer: (a) 20.4 (b) 4.24 (c) 21708

3 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations

$x = t$, $y = t^2$. The line $y = 7x - 12$

meets C at A and B. Find the exact length AB.

Substitute: $t^2 = 7t - 12$.

$t^2 - 7t + 12 = 0 \Rightarrow t = 4$ or $t = 3$.

A(4, 16), B(3, 9).

$AB^2 = 1^2 + 7^2 = 50$.

Answer: AB = $5\sqrt{2}$

4 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = e^x + 2x - 3$.

(a) Show that $f(x) = 0$ has a root α with $0 < \alpha < 1$.

(b) Use Newton-Raphson with $x_0 = 1$ to find x_1 and x_2

to 4 decimal places.

$f(0) = -2.00$ and $f(1) = 1.72$: a change of sign, f continuous

$f'(x) = e^x + 2$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 1 - \frac{1.718}{4.718} = 0.6358$

$x_2 = 0.6358 - \frac{0.1602}{3.889} = 0.5946$

Answer: $x_1 = 0.6358$, $x_2 = 0.5946$

5 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 23 seats. Each row has 3 more seats than the row in front. Find
(a) the number of seats in row 32,
(b) the least number of rows for at least 1632 seats.

Arithmetic: $a = 23$, $d = 3$.
(a) $u_{32} = 23 + 31 \times 3 = 116$.

(b) $S_n = \frac{n}{2}(46 + 3n - 3) = \frac{n(3n + 43)}{2} \geq 1632$.

$S_{26} = 1573 < 1632$, $S_{27} = 1674 \geq 1632$.

Answer: (a) 116 (b) 27 rows

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = t^2$, $y = 2t^3 - 18t$.
Find the coordinates of the points on the curve where the gradient is 6.

$dx/dt = 2t$, $dy/dt = 6t^2 - 18$, so $dy/dx = \frac{6t^2 - 18}{2t}$

Set equal to 6: $6t^2 - 12t - 18 = 0$
$t = -1$ or $t = 3$

Answer: (1, 16) and (9, 0)

7 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 5 kg moves under a single force F N, where $\mathbf{F} = (40t + 25)\mathbf{i} + (30t^2)\mathbf{j}$ and t is the time in seconds. At $t = 0$, P is at the origin with velocity $(-4\mathbf{i} + 4\mathbf{j})$ m s^{-1}.
Find (a) the velocity of P at $t = 3$, (b) the position vector of P at $t = 3$.

$\mathbf{a} = \frac{\mathbf{F}}{5} = (8t + 5)\mathbf{i} + (6t^2)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\, dt = (4t^2 + 5t - 4)\mathbf{i} + (2t^3 + 4)\mathbf{j}$ (using $\mathbf{v}(0) = (-4\mathbf{i} + 4\mathbf{j})$)
At $t = 3$: $\mathbf{v} = (47\mathbf{i} + 58\mathbf{j})$

$\mathbf{r} = \int \mathbf{v}\, dt = (\frac{4}{3}t^3 + 2.5t^2 - 4t)\mathbf{i} + (0.5t^4 + 4t)\mathbf{j}$

At $t = 3$: $\mathbf{r} = (46.5\mathbf{i} + 52.5\mathbf{j})$

Answer: (a) $(47\mathbf{i} + 58\mathbf{j})$ m s^{-1} (b) $(46.5\mathbf{i} + 52.5\mathbf{j})$ m

8 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 6 m and mass 10 kg rests on supports at C and D, where AC = 1 m and AD = 4 m, with $R_C = 3R_D$.
Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 3R_D$. Moments about A: $R_C \times 1 + R_D \times 4 = 10g \times AG$

$R_D(3 \times 1 + 4) = 10g \times AG$ with $R_D(1 + 3) = 10g$, so $AG = \frac{3 \times 1 + 4}{4} = 1.75$

Tilting about D: $R_C = 0$. Moments about D: $mg(6 - 4) = 10g(4 - 1.75)$

$m = \frac{10 \times 2.25}{2} = 11.25$

Answer: (a) 1.75 m (b) 11.25

Day 168 **A Level Maths: S Tier 10-a-day: Model Solutions**

9 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = \frac{2}{x+1}$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (\frac{2}{x+1})^2\,dx$

Simplify the integrand, then integrate

$V = \pi \times (\frac{8}{3})$

Answer: $\frac{8}{3}\pi$

10 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: $p = 0.25$ against H_1: $p > 0.25$ uses a sample of 50 at the 5% level.
(a) Find the critical region for X, the number of successes.
(b) Find the actual significance level.
(c) Find P(Type II error) when in fact $p = 0.45$.

$X \sim B(50, 0.25)$. $P(X \geq 19) = 0.0287$ and $P(X \geq 18) = 0.0551$
The first is at most 0.05 and the second is not, so the critical region is $X \geq 19$
Actual significance level = $P(X \geq 19) = 0.0287$
Type II error: H_0 is not rejected, so $X \leq 18$, with $X \sim B(50, 0.45)$
$P(X \leq 18) = 0.127$
Answer: (a) $X \geq 19$ (b) 0.0287 (c) 0.127