Day 167 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$2^x \times 4^y = 512$

$9^x \div 3^y = 27$

Write each side with prime bases: $512 = 2^9$, $27 = 3^3$.

First: $2^{1x} \times 2^{2y} = 2^9$, so $1x + 2y = 9$.

Second: $3^{2x - 1y} = 3^3$, so $2x - 1y = 3$.

Solving the pair of linear equations: $x = 3$, $y = 3$.

Answer: $x = 3$, $y = 3$

2 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$\int_0^{\pi/3} \tan x \, dx$

$\tan x = \frac{\sin x}{\cos x}$ and $d/dx(\cos x) = -\sin x$

Integral = $[-\ln|\cos x|]$

cos at the upper limit is $\frac{1}{2}$

Answer: ln 2

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(3\mathbf{j})$ N, $(-4\mathbf{i} + k\mathbf{j})$ N and $(-6\mathbf{i} - 4\mathbf{j})$ N act on a particle of mass 5 kg. The resultant is parallel to $(-5\mathbf{i} - \mathbf{j})$. Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(-10)\mathbf{i} + (-1 + k)\mathbf{j}$

Parallel to $(-5\mathbf{i} - \mathbf{j})$: $\frac{-1 + k}{-10} = \frac{-1}{-5}$, so $k = -1$

R = $(-10\mathbf{i} - 2\mathbf{j})$ N, so $|R| = \sqrt{(-10^2 + -2^2)} = 10.20$

a = $|R| \div 5$

Answer: (a) $k = -1$ (b) 2.04 m s^{-2}

4 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

10 recorded masses (grams) have mean 47 and standard deviation 8.

One value was recorded as 32 but should have been 18.

Find the correct mean and the correct standard deviation.

$\Sigma x = 10 \times 47 = 470$, so corrected $\Sigma x = 470 - 32 + 18 = 456$

Correct mean = $\frac{456}{10} = 45.60$

$\Sigma x^2 = 10(64 + 2209) = 22730$, corrected $\Sigma x^2 = 22730 - 1024 + 324 = 22030$

Variance = $\frac{22030}{10} - 45.600^2 = 123.64$, $\sigma = 11.12$

Answer: mean 45.6, standard deviation 11.1

5 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x + 2} - 33 \times 2^x + 8 = 0$.

Let $y = 2^x$. Then $4y^2 - 33y + 8 = 0$.

$4(y - \frac{1}{4})(y - 8) = 0$, so $y = \frac{1}{4}$ or $y = 8$.

$2^x = \frac{1}{4}$ gives $x = -2$; $2^x = 8$ gives $x = 3$.

Answer: $x = -2$ or $x = 3$

6 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 7.2 m, at a horizontal distance of 72 m from O.

Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take g = 9.8 m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$7.2 = 72 - \frac{9.8 \times 72^2}{u^2}$, so $u^2 = \frac{9.8 \times 5184}{64.8} = 784$

u = 28

Greatest height = $\frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{784}{39.2}$

Answer: (a) 28 m s^{-1} (b) 20.0 m

7 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 2$ to find the exact value of

$\int_0^1 x(x^2 + 2)^3\,dx$

du = 2x dx; limits x = 0, 1 become u = 2, 3

Integral = $\frac{1}{2}\int_2^3 u^3\,du = [\frac{u^4}{8}]$

Evaluate the limits

Answer: $\frac{65}{8}$

8 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2y + xy^2 = 84$.

Find the equation of the tangent to C at the point (3, 4),
in the form ax + by + c = 0 with integers a, b, c.

Differentiate implicitly with respect to x

At (3, 4): $dy/dx = -\frac{40}{33}$

$y - 4 = (-\frac{40}{33})(x - 3)$, then clear fractions

Answer: 40x + 33y − 252 = 0

9 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(27 - x)^{\frac{2}{3}}$ in ascending powers of x up to and including the term in x^3.

State the values of x for which it is valid.

$(27 - x)^{\frac{2}{3}} = 27^{\frac{2}{3}}(1 - \frac{1}{27}x)^{\frac{2}{3}} = 9(1 + u)^{\frac{2}{3}}$, $u = -\frac{1}{27}x$.

$(1 + u)^{\frac{2}{3}} = 1 + \frac{2}{3}u + (-\frac{1}{9})u^2 + \frac{4}{81}u^3 + ...$

Substitute u and multiply by 9.

Valid when |u| < 1, i.e. |x| < 27.

Answer: $9 - \frac{2}{9}x - \frac{1}{729}x^2 - \frac{4}{177147}x^3 + ...$; |x| < 27

10 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(9, \frac{1}{4})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{9 - r}{r + 1} \times \frac{1}{3}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(9, r + 1)}{C(9, r)} \times \frac{p}{q} = \frac{9 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{4}$ and $q = \frac{3}{4}$, $p \div q = \frac{1}{3}$

The ratio exceeds 1 when $(9 - r)\frac{1}{4} > (r + 1)\frac{3}{4}$, that is $r < 1.5$

So probabilities rise up to r = 2 and then fall; the mode is 2
P(X = 2) = 0.300

Answer: 2 with probability 0.300