Day 166 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(8, 9) lies on the circle: $(8 - r)^2 + (9 - r)^2 = r^2$.

$r^2 - 34r + 145 = 0$, so r = 5 or r = 29.

The diagram shows P on the larger circle, so r = 29.

$(x - 29)^2 + (y - 29)^2 = 841$, then expand.

Answer: $x^2 + y^2 - 58x - 58y + 841 = 0$

2 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos(3\theta - 45°) = \frac{\sqrt{3}}{2}$

for $0° \leq \theta \leq 360°$.

Let $u = 3\theta - 45°$; then $\cos u = \frac{\sqrt{3}}{2}$

Base solutions: u = 30, −30° (+ multiples of 360°)

θ = (u + 45°) ÷ 3, keeping those in range

Answer: θ = 5°, 25°, 125°, 145°, 245°, 265°

3 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 13 and variance $\frac{26}{3}$.

(a) Find n and p.

(b) Find P(X = 12), correct to 3 significant figures.

$np = 13$ and $np(1 - p) = \frac{26}{3}$

Divide: $1 - p = \frac{26}{3} \div 13 = \frac{2}{3}$, so $p = \frac{1}{3}$

$n = 13 \div \frac{1}{3} = 39$

$P(X = 12) = C(39, 12) (\frac{1}{3})^{12} (\frac{2}{3})^{27} = 0.130$

Answer: (a) n = 39, p = $\frac{1}{3}$ (b) 0.130

4 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0 e^{kt}$, t in hours.

Initially P = 400 and after 5 hours P = 1200.

Find k, and the time for the population to reach 1200, to 3 s.f.

$P_0 = 400$. $400 e^{5k} = 1200$ so $e^{5k} = 3$

$k = \frac{1}{5} \ln 3 = 0.220$

$400 e^{kt} = 1200$ so $e^{kt} = 3$, $t = \frac{1}{k} \ln 3$

t = 5.00 hours

Answer: k = 0.220, t = 5.00 hours

5 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-2t^2 - t)\mathbf{i} + (1.5t^2 + 5t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(\mathbf{i} - \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-4t - 1)\mathbf{i} + (3t + 5)\mathbf{j}$

Parallel to $(\mathbf{i} - \mathbf{j})$: $(-4t - 1) \times -1 = (3t + 5) \times 1$

$t(1) = 4$, so $t = 4$

At $t = 4$: $\mathbf{v} = (-17\mathbf{i} + 17\mathbf{j})$, speed $= \sqrt{(-17^2 + 17^2)}$

Answer: (a) $t = 4$ (b) 24.0 m s^{-1}

6 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A faulty component is present in 0.5% of a population. A test is positive for 98% of those with it and for 8% of those without it.

Find P(positive) and P(has it | positive).

P(positive) = 0.005 × 0.98 + 0.995 × 0.08

= 0.0049 + 0.0796 = 0.0845

$P(\text{has it} \mid \text{positive}) = \frac{0.0049}{0.0845}$

= 0.05799

Answer: (a) 0.0845 (b) 0.0580

7 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of

$y = -2f(\frac{1}{2}x + 1) - 2$.

Inside f: replace x by $\frac{1}{2}x + 1$, so x-coordinates become $(x - 1) \div \frac{1}{2}$.

Outside: y-coordinates become $-2 \times y - 2$.

A: (−14, 4) B: (−6, 6)

C: (−2, 2) D: (2, 6)

Answer: A′(−14, 4), B′(−6, 6), C′(−2, 2), D′(2, 6)

8 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coefficient of x^4 in the expansion of

$(x - 3)(1 + 4x)^6$.

$(1 + 4x)^6$: x^4 term has coefficient $C(6,4) \times 4^4 = 3840$, x^3 term has coefficient $C(6,3) \times 4^3 = 1280$.

x^4 in the product: $(-3) \times 3840 + 1 \times 1280$.

= −10240.

Answer: −10240

9 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(4\mathbf{i} + 4\mathbf{j})$ km relative to a fixed origin and B is at $(3\mathbf{i} - 3\mathbf{j})$ km. A has velocity $(2\mathbf{i} + 3\mathbf{j})$ km h^{-1} and B has velocity $(-\mathbf{i} + 7\mathbf{j})$ km h^{-1}.

Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-\mathbf{i} - 7\mathbf{j}) + t(-3\mathbf{i} + 4\mathbf{j})$

$|\mathbf{d}|^2 = (-3t - 1)^2 + (4t - 7)^2 = 25t^2 - 50t + 50$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-25}{25} = 1$

$\mathbf{d} = (-\mathbf{i} - 7\mathbf{j}) + 1(-3\mathbf{i} + 4\mathbf{j}) = (-4\mathbf{i} - 3\mathbf{j})$, so least distance = 5

Answer: (a) $t = 1$ h (b) 5 km

10 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3t^2$, $y = 3t(6 - t)$,
for $0 \leq t \leq 6$. The curve meets the x-axis at O and at A.
Find the area enclosed between the curve and the x-axis.

$dx/dt = 6t$; x from 0 to 108 as t goes from 0 to 6

Area $= \int_0^6 y \frac{dx}{dt}\, dt = \int_0^6 18t^2(6 - t)\, dt$

$= 18[\frac{6t^3}{3} - \frac{t^4}{4}]$

Answer: 1944