Day 165 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A pile of logs has 8 logs in its bottom row. Each row above has 1 fewer logs than the row below, and the pile has as many rows as possible, each with at least one log. Find the number of rows and the total number of logs.

Row n has $8 - 1(n - 1) = -n + 9$ logs; need this ≥ 1.
$1(n - 1) \leq 7$, so $n \leq 8$: 8 rows.

Top row = 1 logs. $S = \frac{8}{2}(8 + 1) = 36$.

Answer: 8 rows, 36 logs

2 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 10) + \log_2(x - 2) = 6$,
justifying clearly any root that you reject.

$(x + 10)(x - 2) = 2^6 = 64$
$x^2 + 8x - 84 = 0$
$(x - 6)(x + 14) = 0$, so $x = 6$ or $x = -14$
$x = -14$ makes $x + 10 = -4 < 0$ and log of a negative number is undefined
Answer: $x = 6$

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-4 - 3\sqrt{7}}{3 + \sqrt{7}}$ in the form $p + q\sqrt{7}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(3 - \sqrt{7})$.
Numerator: $(-4 - 3\sqrt{7})(3 - \sqrt{7}) = 9 - 5\sqrt{7}$.
Denominator: $3^2 - 1^2(7) = 2$.
So the fraction = $\frac{9 - 5\sqrt{7}}{2}$.

Answer: $\frac{9}{2} - \frac{5}{2}\sqrt{7}$

4 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are normally distributed. 30% are less than 45 and 5% are greater than 76.
Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 45) = 0.30$ gives $\frac{45 - \mu}{\sigma} = -0.5244$

$P(X > 76) = 0.05$ gives $\frac{76 - \mu}{\sigma} = 1.6449$

Subtract: $\frac{31}{\sigma} = 1.6449 - (-0.5244) = 2.1693$

$\sigma = 14.29$
$\mu = 45 - (-0.5244)(14.29) = 52.49$
Answer: $\mu = 52.5$, $\sigma = 14.3$

5 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from level ground with horizontal and vertical velocity components 34.3 and 49 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take $g = 9.8$ m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 34.3$
Moving upwards: $49 - 9.8t = 34.3$, so $t = 1.5$
Moving downwards: $49 - 9.8t = -34.3$, so $t = 8.5$
Horizontal distance = $34.3 \times (8.5 - 1.5)$
Answer: (a) 1.5 s and 8.5 s (b) 240.1 m

6 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $4\sec^2\theta + 7\tan\theta - 6 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $4\tan^2\theta + 7\tan\theta - 2 = 0$
$(4\tan\theta - 1)(\tan\theta + 2) = 0$

$\tan\theta = \frac{1}{4}$ or $\tan\theta = -2$

$\theta = 14.0°$ and $116.6°$ $(+180°)$
Answer: θ = 14.0°, 116.6°, 194.0°, 296.6°

7 Poisson hypothesis test (CIE)

Faults in a cable occur at random at a mean rate of 5 per week. After a modification, 19 faults are found in 2 weeks. Test at the 5% level whether the rate of faults has increased.

H_0: $\lambda = 5$ per week, H_1: $\lambda > 5$ per week
Over 2 weeks: $X \sim Po(10)$
$P(X \geq 19) = 1 - P(X \leq 18) = 0.00719$
$0.00719 < 5\%$, so reject H_0
There is sufficient evidence that the rate has increased
Answer: Reject H_0 (p-value 0.00719)

8 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 5x + 14$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(5x + 14)}$.
(b) Use $x_{n+1} = \sqrt[3]{(5x_n + 14)}$ with $x_0 = 3$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(5x + 14)}$
$x_1 = 3.0723$, $x_2 = 3.0850$, $x_3 = 3.0873$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 3.088$
Answer: x_1 = 3.0723, x_2 = 3.0850, x_3 = 3.0873; α = 3.088

9 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = t$, $y = 4t^2$. The line $y = -20x - 24$
meets C at A and B. Find the exact length AB.

Substitute: $4t^2 = -20t - 24$.
$4t^2 + 20t + 24 = 0 \Rightarrow t = -3$ or $t = -2$.
A(−3, 36), B(−2, 16).
$AB^2 = (-1)^2 + 20^2 = 401$.
Answer: AB = $\sqrt{401}$

10 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 12 kg and 9 kg hang on a light string over a smooth fixed pulley. The 12 kg particle is 1.75 m above the ground; the 9 kg one is on the ground.
Released from rest, find (a) the tension, (b) the greatest height reached by the 9 kg particle. Take $g = 9.8$ m s^{-2}.

$12g - T = 12a$ and $T - 9g = 9a$, so $a = \frac{12 - 9}{12 + 9} \times 9.8 = 1.4$

$T = 12(9.8 - 1.4) = 100.8$
The 9 kg particle rises 1.75 m before the 12 kg particle lands, reaching $v^2 = 2 \times 1.4 \times 1.75 = 4.9$
The string goes slack, so it then moves freely under gravity: extra height $= \frac{v^2}{2g} = \frac{4.9}{19.6} = 0.25$
Greatest height = 1.75 + 0.25
Answer: (a) 100.8 N (b) 2 m