Day 164 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations

$\log_5 x - \log_5 y = 1$

$x + y = 6$

$\log_5(x/y) = 1$, so $x = 5^1 y = 5y$

$5y + y = 6$, so $6y = 6$

Answer: x = 5, y = 1

2 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 21 m s^{-1} from the top of a cliff. The top of the cliff is 50.4 m above the sea.

Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-50.4 = 21t - 4.9t^2$, so $4.9t^2 - 21t - 50.4 = 0$

$$t = \frac{21 + \sqrt{(21^2 + 19.6 \times 50.4)}}{9.8} = \frac{21 + 37.8}{9.8} = 6$$

Height risen above the cliff top = $\frac{21^2}{(2 \times 9.8)} = 22.5$ m

Total distance = $2 \times 22.5 + 50.4$

Answer: (a) 6 s (b) 95.4 m

3 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 19 seats. Each row has 3 more seats than the row in front. Find

(a) the number of seats in row 34,

(b) the least number of rows for at least 1265 seats.

Arithmetic: a = 19, d = 3.

(a) $u_{34} = 19 + 33 \times 3 = 118$.

(b) $S_n = \frac{n}{2}(38 + 3n - 3) = \frac{n(3n + 35)}{2} \geq 1265$.

$S_{23} = 1196 < 1265$, $S_{24} = 1284 \geq 1265$.

Answer: (a) 118 (b) 24 rows

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $4\sec^2\theta - 9\tan\theta - 2 = 0$

for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $4\tan^2\theta - 9\tan\theta + 2 = 0$

$(\tan\theta - 2)(4\tan\theta - 1) = 0$

$\tan\theta = 2$ or $\tan\theta = \frac{1}{4}$

$\theta = 63.4°$ and $14.0°$ $(+180°)$

Answer: θ = 14.0°, 63.4°, 194.0°, 243.4°

5 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x - y = 1$

$x^2 - xy + y^2 = 1$

From the first equation $x = 1 + y$.

Substitute into the second equation and collect terms:

this gives $y^2 + y = 0$.

$y = -1$ or $y = 0$, giving $x = 0$ or $x = 1$.

Answer: (x, y) = (0, −1) and (1, 0)

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = x^4e^{-3x}$ for $x > 0$,
and determine its nature.

$dy/dx = x^3e^{-3x}(4 - 3x) = 0$ gives $x = 0$ or $x = \frac{4}{3}$

$y = \frac{256}{81}e^{-4}$ when $x = \frac{4}{3}$

Second derivative at this point is negative, so it is a maximum

Answer: $(\frac{4}{3}, \frac{256}{81}e^{-4})$, maximum

7 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 1, 2, 3, 4 with $P(X = x) = k(x + 1)$. X_1 and X_2 are independent copies of X.
(a) Find k.
(b) Find $P(X_1 + X_2 = 4)$.
(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(2 + 3 + 4 + 5) = 14$ $k = 1$, so $k = \frac{1}{14}$

(b) Ways to total 4: (1,3), (2,2), (3,1)

$P = \frac{1}{7}\times\frac{2}{7} + \frac{3}{14}\times\frac{3}{14} + \frac{2}{7}\times\frac{1}{7} = \frac{25}{196}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{27}{98}$

$E(X) = \frac{20}{7}$, so $E(X_1 + X_2) = 2 \times \frac{20}{7} = \frac{40}{7}$

Answer: (a) $k = \frac{1}{14}$ (b) $\frac{25}{196}$ (c) $\frac{27}{98}$, $\frac{40}{7}$

8 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations
$x = 2(t + \frac{1}{t})$, $y = 3(t - \frac{1}{t})$, $t \neq 0$.
Find a Cartesian equation of the curve.

$x^2 = 2^2(t^2 + 2 + \frac{1}{t^2})$.

$y^2 = 3^2(t^2 - 2 + \frac{1}{t^2})$.

$\frac{x^2}{4} - \frac{y^2}{9} = 4$.

Answer: $\frac{x^2}{4} - \frac{y^2}{9} = 4$

9 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 21 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 39 m horizontally and 2 m vertically from O.
Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 21\cos\theta \times t$. Vertical: $y = 21\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 39^2}{2 \times 21^2} = 16.9$, so $2 = 39\tan\theta - 16.9(1 + \tan^2\theta)$

$16.9\tan^2\theta - 39\tan\theta + 18.9 = 0$

$\tan\theta = \frac{9}{13}$ or $\frac{21}{13}$, so $\theta = 34.7°$ or $58.2°$

Answer: $\tan\theta = \frac{9}{13}$ or $\frac{21}{13}$; $\theta = 34.7°$ or $58.2°$

10 Pdf of the form a + bx with a given mean (CIE)

The pdf of X is f(x) = a + bx for $0 \leq x \leq 3$, and 0 otherwise, where a and b are constants. E(X) = $\frac{9}{5}$.

(a) Find a and b.
(b) Find P(X < 1.5).

Total area = 1: 3 a + 4.5 b = 1

E(X) = $\int x(a + bx)\,dx$ = 4.5 a + 9 b = $\frac{9}{5}$

Solving the simultaneous equations: a = $\frac{2}{15}$, b = $\frac{2}{15}$

P(X < 1.5) = $\int f(x)\,dx$ from 0 to 1.5 = $\frac{2}{15} \times 1.5 + \frac{2}{15} \times 1.125$

= $\frac{7}{20}$

Answer: (a) a = $\frac{2}{15}$, b = $\frac{2}{15}$ (b) $\frac{7}{20}$