Day 160 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 2 green counters and some black counters, n in total.

Two are taken without replacement. P(at least one green) = $\frac{5}{6}$. Find n.

P(neither green) = $\frac{(n-2)}{n} \times \frac{(n-3)}{(n-1)} = \frac{1}{6}$

$\frac{1}{6}\, n(n-1) = (n-2)(n-3)$

Rearranging gives a quadratic in n whose roots are n = 4 and one other value

The other root is 1.8 so is rejected (not a valid number of counters)

n = 4

Answer: n = 4

2 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\tan(3\theta - 45°) = -\frac{1}{\sqrt{3}}$

for $0° \le \theta \le 360°$.

Let $u = 3\theta - 45°$; then $\tan u = -\frac{1}{\sqrt{3}}$

Base solutions: $u = -30°$ (+ multiples of 180°)

$\theta = (u + 45°) \div 3$, keeping those in range

Answer: θ = 5°, 65°, 125°, 185°, 245°, 305°

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Show that $\frac{7+\sqrt{6}}{7-\sqrt{6}} - \frac{7-\sqrt{6}}{7+\sqrt{6}}$

can be written in the form $k\sqrt{6}$, and find k.

Common denominator $(7-\sqrt{6})(7+\sqrt{6}) = 49 - 6 = 43$.

Numerator: $(7+\sqrt{6})^2 - (7-\sqrt{6})^2$

$= (49 + 6 + 27\sqrt{6}) - (49 + 6 - 27\sqrt{6}) = 28\sqrt{6}$.

So the expression = $\frac{28\sqrt{6}}{43}$.

Answer: k = $\frac{28}{43}$

4 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 25 observations x, $\Sigma(x - 50) = 5$ and $\Sigma(x - 50)^2 = 351$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 50) = \frac{5}{25} = 0.2$, so mean of x = 50.2

Variance = $\frac{351}{25} - (0.2)^2 = 14$

$\sigma = \sqrt{14}$

$\Sigma x^2 = \Sigma(x - 50)^2 + 100\,\Sigma(x - 50) + 25 \times 50^2$

$= 351 + 500 + 62500$

Answer: (a) 50.2 (b) 3.74 (c) 63351

5 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the term independent of x in the expansion of $(x + \frac{1}{x})^{10}$.

General term: $C(10,r)(x)^{10-r}(\frac{1}{x})^r$.

Power of x: $1(10 - r) - 1r = 0$, so $r = 5$.

Term = $C(10,5) \times 1^5 \times (1)^5 = 252$.

Answer: 252

6 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from level ground with horizontal and vertical velocity components 29.4 and 53.9 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, |v_y| = v_x = 29.4

Moving upwards: $53.9 - 9.8t = 29.4$, so $t = 2.5$

Moving downwards: $53.9 - 9.8t = -29.4$, so $t = 8.5$

Horizontal distance = $29.4 \times (8.5 - 2.5)$

Answer: (a) 2.5 s and 8.5 s (b) 176.4 m

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3x^2\sqrt{5 - x}$, for $0 < x < 5$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{3x(20 - 5x)}{2\sqrt{5 - x}}$

$\frac{dy}{dx} = 0$: $20 - 5x = 0$, so $x = 4$

$y = 3 \times 4^2 \times 1$

Gradient changes from positive to negative: a maximum

Answer: (4, 48), maximum

8 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -5 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$.

The point P has coordinates (−3, 5, 1).
Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} - t\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$

FP · d = 0: $24 - t(12) = 0$, so $t = 2$

F = (−1, 2, 0)

$\mathrm{PF} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{14}$

Answer: F = (−1, 2, 0), PF = $\sqrt{14}$

9 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 40 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{7}{24}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{3}{4}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 40\cos\alpha$, so $R = 40 \times \frac{24}{25} - P\sin\beta$

Along the plane: $P\cos\beta = 40\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 40(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{4}{5} + \frac{3}{4} \times \frac{3}{5}) = 40(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25})$, so $P = 32$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_min = 40(\frac{7}{25} + \frac{3}{4} \times \frac{24}{25}) \div \frac{5}{4}$

Answer: (a) 32 N (b) 32 N

10 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 2 and common ratio 5.

Find the least n for which $S_n > 59544$.

$S_n = \frac{2(5^n - 1)}{4} > 59544$.

$5^n > 119089$.

$n > \frac{\log(1.191e+05)}{\log(5)} = 7.262$.

So the least n is 8.

Answer: n = 8