Day 159 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 4 = 0$.

Let $u = x^{\frac{1}{3}}$: $2u^2 + 6u + 4 = 0$.
$(u + 2)(u + 1) = 0$, so $u = -2$ or $u = -1$.
Cube: $x = -8$ or $x = -1$.
Answer: $x = -8$ or $x = -1$

2 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 5) + \log_2(x - 2) = 3$,
justifying clearly any root that you reject.

$(x + 5)(x - 2) = 2^3 = 8$
$x^2 + 3x - 18 = 0$
$(x - 3)(x + 6) = 0$, so $x = 3$ or $x = -6$
$x = -6$ makes $x + 5 = -1 < 0$ and log of a negative number is undefined
Answer: $x = 3$

3 Waiting for the r-th success (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each shot Amira takes scores with probability $\frac{3}{5}$, independently.

(a) Find the probability that her fourth score comes on shot number 6.
(b) Find the probability that her fourth score comes on or before shot number 6.

(a) fourth score on shot 6: exactly 3 scores in the first 5 shots, then a score
$C(5, 3)(\frac{3}{5})^3(\frac{2}{5})^2 \times \frac{3}{5}$
$= 0.207$

(b) On or before shot 6 means at least 4 scores in 6 shots: $X \sim B(6, \frac{3}{5})$
$P(X \geq 4) = 1 - P(X \leq 3) = 0.544$
Answer: (a) 0.207 (b) 0.544

4 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 80 m wide and flows at 12 m s^{-1}. A boatman can row at 13 m s^{-1} in still water and wants to reach the point directly opposite his starting point.
Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current
Let the boat head at angle φ upstream from the perpendicular: $13 \sin \varphi = 12$, so $\varphi = 67.4°$
Angle to the bank = $90° - \varphi = 22.6°$
Speed across = $\sqrt{(13^2 - 12^2)} = 5$ m s^{-1}
Time = $\frac{80}{5} = 16.0$ s
Answer: (a) 22.6° (b) 16.0 s

5 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 5x^2 + 12x - 18$.
Show that $(x - 3)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(3) = 27 - 45 + 36 - 18 = 0$, so $(x - 3)$ is a factor.
$f(x) = (x - 3)(x^2 - 2x + 6)$.
For $x^2 - 2x + 6 = 0$: $b^2 - 4ac = 4 - 24 = -20 < 0$.
No real roots from the quadratic, so the only real root is $x = 3$.
Answer: Proved; the only real root is $x = 3$

6 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 20 observations x, $\Sigma(x - 50) = 30$ and $\Sigma(x - 50)^2 = 265$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 50) = \frac{30}{20} = 1.5$, so mean of $x = 51.5$

Variance $= \frac{265}{20} - (1.5)^2 = 11$

$\sigma = \sqrt{11}$

$\Sigma x^2 = \Sigma(x - 50)^2 + 100\,\Sigma(x - 50) + 20 \times 50^2$

$= 265 + 3000 + 50000$

Answer: (a) 51.5 (b) 3.32 (c) 53265

7 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 8 m and mass 5 kg rests on supports at C and D, where AC = 1 m and AD = 5 m, with $R_C = 1.5\,R_D$.
Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D.
(b) Find m.

$R_C = 1.5\,R_D$. Moments about A: $R_C \times 1 + R_D \times 5 = 5g \times AG$

$R_D(1.5 \times 1 + 5) = 5g \times AG$ with $R_D(1 + 1.5) = 5g$, so $AG = \frac{1.5 \times 1 + 5}{2.5} = 2.6$

Tilting about D: $R_C = 0$. Moments about D: $m g (8 - 5) = 5 g (5 - 2.6)$

$m = \frac{5 \times 2.4}{3} = 4$

Answer: (a) 2.6 m (b) 4

8 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = e^{2x}$, the x-axis and the lines
$x = 0$ and $x = 1$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 (e^{2x})^2\,dx$

Simplify the integrand, then integrate

$V = \pi \times (-\frac{1}{4} + \frac{1}{4}e^4)$

Answer: $\frac{1}{4}\pi(-1 + e^4)$

9 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x + y = 1$
$3x^2 + y^2 = 13$

From the first equation $x = 1 - y$.
Substitute into the second equation and collect terms:
this gives $4y^2 - 6y - 10 = 0$.
$y = -1$ or $y = \frac{5}{2}$, giving $x = 2$ or $x = -\frac{3}{2}$.

Answer: (x, y) = (2, −1) and $(-\frac{3}{2}, \frac{5}{2})$

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10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 11 \\ -10 \\ -2 \end{pmatrix} + s\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -4 \\ 0 \\ -11 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $11 - 2s = -4 + 3t$; $-10 + 3s = 0$; $-2 = -11 + 3t$

First two give $s = 10/3$, $t = 25/9$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \frac{|-6|}{\sqrt{13}\sqrt{18}} = 0.3922$, $\varphi = 66.9°$

Answer: Lines do not meet (skew); angle 66.9°