Day 158 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 5 m and mass 6 kg rests on supports at C and D, where AC = 1 m and AD = 4 m, with R_C = 3 R_D.
Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

R_C = 3 R_D. Moments about A: R_C × 1 + R_D × 4 = 6 g × AG

R_D(3 × 1 + 4) = 6 g × AG with R_D(1 + 3) = 6 g, so AG = $\frac{3 \times 1 + 4}{4}$ = 1.75

Tilting about D: R_C = 0. Moments about D: m g (5 − 4) = 6 g (4 − 1.75)

m = $\frac{6 \times 2.25}{1}$ = 13.5

Answer: (a) 1.75 m (b) 13.5

2 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0 e^{kt}$, t in hours.
Initially P = 50 and after 2 hours P = 80.
Find k, and the time for the population to reach 500, to 3 s.f.

P_0 = 50. 50 e^{2k} = 80 so $e^{2k} = \frac{8}{5}$

k = $\frac{1}{2}$ ln $\frac{8}{5}$ = 0.235

50 e^{kt} = 500 so e^{kt} = 10, t = $\frac{1}{k}$ ln 10

t = 9.80 hours

Answer: k = 0.235, t = 9.80 hours

3 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 - 2x - 15 \leq 0$ and $-x - 5 < -5$.

Quadratic: (x + 3)(x − 5) ≤ 0 gives −3 ≤ x ≤ 5.
Linear: −x − 5 < −5 gives x > 0. (the inequality reverses as we divide by a negative)
Both must hold: take the overlap of the two sets.
Answer: 0 < x ≤ 5

4 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces (−3**j**) N, (6**i** + k**j**) N and (−3**i** + 6**j**) N act on a particle of mass 5 kg. The resultant is parallel to (3**i** + 4**j**).
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = (3)**i** + (3 + k)**j**

Parallel to (3**i** + 4**j**): $\frac{3 + k}{3} = \frac{4}{3}$, so k = 1

R = (3**i** + 4**j**) N, so |R| = $\sqrt{(3^2 + 4^2)}$ = 5.000
a = |R| ÷ 5

Answer: (a) k = 1 (b) 1.00 m s^{-2}

5 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = x\sqrt{4x + 12}$ at the point where $x = 1$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 1$: $y = 4$. Differentiate: gradient $= \frac{9}{2}$

Tangent: $y - 4 = \frac{9}{2}(x - 1)$, so $y = \frac{9}{2}x - \frac{1}{2}$

y-intercept $-\frac{1}{2}$; x-intercept $\frac{1}{9}$

Area $= \frac{1}{2} \times |-\frac{1}{2}| \times |\frac{1}{9}|$

Answer: $\mathbf{\frac{1}{36}}$

6 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A manufacturer says that 20% of its screens have no flaws. A sample of 25 screens is checked, and 7 have no flaws. Test, at the 1% significance level, the claim that the proportion is higher. Give your conclusion in context.

H_0: $p = 0.2$, H_1: $p > 0.2$. Let $X \sim B(25, 0.2)$

$P(X \geq 7) = 1 - P(X \leq 6) = 0.220$

0.220 is greater than 1%, so do not reject H_0

There is insufficient evidence that the proportion is higher than the claimed 0.2

Answer: Do not reject H_0: p-value 0.220 > 1%

7 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx - 4$, where a and b are constants. $(x + 2)$ is a factor of f(x), and f(x) leaves remainder 80 when divided by $(x - 3)$. Find a and b, and hence solve $f(x) = 0$.

$f(-2) = 0$: $-16 + 4a - 2b - 4 = 0$, so $4a - 2b - 20 = 0$.

$f(3) = 80$: $9a + 3b + 50 = 80$.

Solving: $a = 4$, $b = -2$.

$f(x) = 2x^3 + 4x^2 - 2x - 4 = 2(x + 2)(x + 1)(x - 1)$

Answer: a = 4, b = −2; x = −2, −1, 1

8 Sums of inverse tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B are acute angles with $\tan A = \frac{1}{12}$ and $\tan B = \frac{11}{13}$.

Find $\tan(A + B)$ and hence show that $A + B = \frac{\pi}{4}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{12} + \frac{11}{13}}{1 - \frac{11}{156}} = \frac{\frac{145}{156}}{\frac{145}{156}} = 1$$

$A + B$ is acute, so $A + B = \pi/4$

Answer: $\tan(A + B) = 1$, $A + B = \frac{\pi}{4}$

9 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 7 and common ratio $\frac{4}{5}$.

Find the least n for which S_n exceeds 99% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.99\, S_\infty$.

$1 - \frac{4}{5}^n > 0.99$, so $\frac{4}{5}^n < 0.01$.

$n > \frac{\log(0.01)}{\log(0.8)} = 20.638$.

So the least n is 21.

Answer: n = 21

10 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(28, \frac{1}{5})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{28 - r}{r + 1} \times \frac{1}{4}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(28, r + 1)}{C(28, r)} \times \frac{p}{q} = \frac{28 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{5}$ and $q = \frac{4}{5}$, $p \div q = \frac{1}{4}$

The ratio exceeds 1 when $(28 - r)\frac{1}{5} > (r + 1)\frac{4}{5}$, that is $r < 4.8$

So probabilities rise up to r = 5 and then fall; the mode is 5

P(X = 5) = 0.186

Answer: 5 with probability 0.186