Day 157 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{3}{14 + 12\sin\theta + 5\cos\theta}$

(a) Write $12\sin\theta + 5\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{12^2 + 5^2} = 13$, $\tan\alpha = \frac{5}{12}$, $\alpha = 22.6°$

Denominator $= 14 + 13\sin(\theta + 22.6°)$, which lies between 1 and 27

Greatest $f(\theta) = \frac{3}{1} = 3$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{3}{27} = \frac{1}{9}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 22.6° = 270°$, so $\theta = 247.4°$

Answer: (a) $13\sin(\theta + 22.6°)$ (b) greatest 3, least $\frac{1}{9}$, $\theta = 247.4°$

2 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 3 m. After each bounce it rises to $\frac{1}{2}$ of its previous height.

(a) Find the total distance it travels before it stops.

(b) After how many bounces does it rise less than 0.01 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 3 + \frac{2 \times 3 \times \frac{1}{2}}{1 - \frac{1}{2}} = 9.$

(b) Rebound height after n bounces $= 3(\frac{1}{2})^n < 0.01$.

$n > \frac{\log(0.00333333)}{\log(0.5)} = 8.23$, so $n = 9$.

Answer: (a) 9 m (b) 9 bounces

3 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 9 m s^{-1} from the top of a cliff. The top of the cliff is 122.4 m above the sea.

Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take $g = 9.8$ m s^{-2}.

Take upwards as positive: $-122.4 = 9t - 4.9t^2$, so $4.9t^2 - 9t - 122.4 = 0$

$t = \frac{9 + \sqrt{(9^2 + 19.6 \times 122.4)}}{9.8} = \frac{9 + 49.8}{9.8} = 6$

Height risen above the cliff top $= \frac{9^2}{(2 \times 9.8)} = 4.13$ m

Total distance $= 2 \times 4.13 + 122.4$

Answer: (a) 6 s (b) 131 m

4 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(60, 0.2)$.

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate $P(X \leq 12)$, to 3 significant figures.

$np = 12$ and $nq = 48$, both greater than 5 (n large, p not extreme)

$Y \sim N(12, 9.6)$ with $\sigma = 3.098$

Apply the continuity correction:

$P(X \leq 12) \approx P(Y < 12.5) = P(Z < 0.161) = 0.564$

Answer: 0.564

5 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $e^x = 3x + 4$ has a root α.
(a) Show that the equation can be written as $x = \ln(3x + 4)$.
(b) Use $x_{n+1} = \ln(3x_n + 4)$ with $x_0 = 1$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take natural logarithms of both sides to obtain $x = \ln(3x + 4)$
$x_1 = 1.9459$, $x_2 = 2.2862$, $x_3 = 2.3850$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.422$
Answer: $x_1 = 1.9459$, $x_2 = 2.2862$, $x_3 = 2.3850$; $\alpha = 2.422$

6 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 10 m and mass 30 kg rests on supports at C and D, where AC = 3 m and DB = 3 m. A girl of mass 50 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(2) = 50(x - 7)$, so $x = 7 + \frac{30 \times 2}{50} = 8.2$

Tilting about C (girl before C): $30(2) = 50(3 - x)$, so $x = 3 - \frac{30 \times 2}{50} = 1.8$

Answer: $1.8 \leq x \leq 8.2$

7 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$P(X = x) = kx(6 - x)$ for $x = 1, 2, ..., 5$, and 0 otherwise.

(a) Show that $k = \frac{1}{35}$.
(b) Find E(X) and Var(X).
(c) Find Var(2X + 4).

(a) $\Sigma P(X = x) = k(5 + 8 + 9 + 8 + 5) = 35k = 1$, so $k = \frac{1}{35}$

(b) $E(X) = \Sigma x P(X = x) = \frac{1\times5 + 2\times8 + 3\times9 + 4\times8 + 5\times5}{35} = 3$

$E(X^2) = \frac{1\times5 + 4\times8 + 9\times9 + 16\times8 + 25\times5}{35} = \frac{53}{5}$

$Var(X) = \frac{53}{5} - 3^2 = \frac{8}{5}$

(c) $Var(2X + 4) = 2^2 Var(X) = 4 \times \frac{8}{5} = \frac{32}{5}$

Answer: (b) $E(X) = 3$, $Var(X) = \frac{8}{5}$ (c) $\frac{32}{5}$

8 Geometric series: sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has sum to infinity 25 and second term 4. Find the possible values of the common ratio r and the first term a in each case.

$a = \frac{4}{r}$. $S_\infty = \frac{a}{1 - r} = \frac{4}{r(1 - r)} = 25$.

$25r^2 - 25r + 4 = 0$.

$(5r - 1)(5r - 4) = 0$: $r = \frac{1}{5}$ or $r = \frac{4}{5}$.

Need $|r| < 1$: $\frac{1}{5}$, $\frac{4}{5}$ valid.

Answer: $r = \frac{1}{5}$, $a = 20$ or $r = \frac{4}{5}$, $a = 5$

Day 157 **A Level Maths: S Tier 10-a-day: Model Solutions**

9 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = \frac{6}{x + 1}$, the x-axis and the lines $x = 0$ and $x = 1$ is rotated through 2π about the x-axis.
Find the exact volume of the solid formed.

$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 \left(\frac{6}{x + 1}\right)^2 dx$

Simplify the integrand, then integrate

$V = \pi \times (18)$

Answer: 18π

10 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 9 and common ratio $\frac{4}{5}$.

Find the least n for which S_n exceeds 99% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.99\ S_\infty$.

$1 - \left(\frac{4}{5}\right)^n > 0.99$, so $\left(\frac{4}{5}\right)^n < 0.01$.

$n > \frac{\log(0.01)}{\log(0.8)} = 20.638$.

So the least n is 21.

Answer: n = 21