Day 156 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(63, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find $P(X > 85)$.

The area 0.9 leaves 0.100 in total in the tails, 0.050 in each

$P(X < 48) = 0.050$ so $\frac{48 - 63}{\sigma} = -1.6449$

$\sigma = \frac{15}{1.6449} = 9.119$

$P(X > 85) = P(Z > 2.412) = 0.00792$

Answer: (a) 9.12 (b) 0.00792

2 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 - 2$. Also $fg(x) = -2x^2 - 2$, where f is a linear function. (a) Find f(x).
(b) Solve $gf(x) = 142$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 - 2) + b = -2x^2 - 2$.
So $a = -2$ and $4 + b = -2$, giving $b = -6$.
$f(x) = -2x - 6$.

(b) $gf(x) = (-2x - 6)^2 - 2 = 142$.

$(-2x - 6)^2 = 144$, so $-2x - 6 = \pm 12$.

$x = -9$ or $x = 3$.

Answer: (a) f(x) = −2x − 6 (b) x = −9 or x = 3

3 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2^x - 2x$ has one stationary point.
Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 2^x \ln 2 - 2 = 0$

$2^x = \frac{2}{\ln 2}$, $x = \frac{1}{\ln 2} \ln(\frac{2}{\ln 2}) = 1.53$

$d^2y/dx^2 = 2^x(\ln 2)^2 > 0$ for all x: minimum

$y = \frac{2}{\ln 2} - 2 \times x = -0.172$

Answer: (1.53, −0.172)

4 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, $P(A) = 0.5$, $P(B) = 0.15$ and $P(A \mid B) = 0.3$.
(a) Find $P(A \cap B)$.
(b) Find $P(A \mid B')$.

$P(A \cap B) = P(A \mid B) \times P(B) = 0.3 \times 0.15 = 0.045$
$P(A \cup B) = 0.5 + 0.15 - 0.045 = 0.605$
$P(B') = 0.85$
$P(A \cap B') = P(A) - P(A \cap B) = 0.455$
$P(A \mid B') = \frac{0.455}{0.85}$

Answer: (a) 0.045 (b) $\frac{91}{170}$

5 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 80 kg, hangs from a vertical cable. The tension in the cable is 3922 N and the floor exerts a force of 592 N on the passenger.

Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take g = 9.8 m s^{-2}.

Passenger (upwards positive): $592 - 80 \times 9.8 = 80\,a$, so $a = -2.4$

$a = 2.4$ m s^{-2}, so the acceleration is downwards

Lift and passenger: $3922 - (M + 80) \times 9.8 = (M + 80) \times -2.4$

$(M + 80)(9.8 - 2.4) = 3922$, so $M + 80 = \frac{3922}{7.4} = 530$

Answer: (a) 2.4 m s^{-2} downwards (b) M = 450

6 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-6 - \sqrt{7}}{4 - \sqrt{7}}$ in the form $p + q\sqrt{7}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 + \sqrt{7})$.

Numerator: $(-6 - \sqrt{7})(4 + \sqrt{7}) = -31 - 10\sqrt{7}$.

Denominator: $4^2 - 1^2(7) = 9$.

So the fraction = $\frac{-31 - 10\sqrt{7}}{9}$.

Answer: $-\frac{31}{9} - \frac{10}{9}\sqrt{7}$

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2\sqrt{20 - x}$, for $0 < x < 20$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{x(80 - 5x)}{2\sqrt{20 - x}}$

$\frac{dy}{dx} = 0$: $80 - 5x = 0$, so $x = 16$

$y = 16^2 \times 2$

Gradient changes from positive to negative: a maximum

Answer: (16, 512), maximum

8 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 18 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 18 m horizontally and 5 m vertically from O.

Find the two possible values of tan θ, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 18 \cos\theta \times t$. Vertical: $y = 18 \sin\theta \times t - 4.9t^2$

Eliminating t: $y = x \tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 18^2}{2 \times 18^2} = 4.9$, so $5 = 18 \tan\theta - 4.9(1 + \tan^2\theta)$

$4.9 \tan^2\theta - 18 \tan\theta + 9.9 = 0$

$\tan\theta = \frac{33}{49}$ or 3, so $\theta = 34.0°$ or $71.6°$

Answer: $\tan\theta = \frac{33}{49}$ or 3; $\theta = 34.0°$ or $71.6°$

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

P is a point on the curve $y^2 = x$. Find the exact minimum distance from P to the point (7, 0), and the x coordinate of P at which it occurs.

$d^2 = (x - 7)^2 + x = x^2 - 13x + 49$

$d(d^2)/dx = 2x - 13 = 0$, so $x = \frac{13}{2}$

Second derivative is $2 > 0$: minimum

$d^2 = \frac{27}{4}$

Answer: distance = $\frac{3\sqrt{3}}{2}$, x = $\frac{13}{2}$

10 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 4}{x - 3}$ for $x \neq 3$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 3) = kx + 4$, so $x(y - k) = 3y + 4$.

$f^{-1}(x) = \frac{3x + 4}{x - k}$.

$f = f^{-1}$ requires $k = 3$.

$f(x) = x$: $3x + 4 = x(x - 3)$, so $x^2 - 6x - 4 = 0$.

$x = 3 \pm \sqrt{13} = 3 \pm \sqrt{13}$.

Answer: k = 3; x = $3 \pm \sqrt{13}$