Day 154 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Slide down a rough slope then along rough ground (CIE)

A particle starts from rest at the top of a rough slope of length 10 m ($\sin\alpha = \frac{3}{5}$, $\mu = 0.3$), then slides along rough horizontal ground with the same μ.

Use energy to find (a) its speed at the foot of the slope (3 s.f.), (b) the distance it slides on the ground. Take $g = 9.8$ m s^{-2}.

On the slope: $\frac{1}{2}mv^2 = mg \times 10\sin\alpha - \mu mg\cos\alpha \times 10$

$v^2 = 2 \times 9.8 \times 10(\frac{3}{5} - 0.3 \times \frac{4}{5}) = 70.56$, so $v = 8.400$

On the ground: $\frac{1}{2}mv^2 = \mu mgx$, so $x = \frac{v^2}{2\mu g} = \frac{70.56}{5.88} = 12$

Answer: (a) 8.40 m s^{-1} (b) 12 m

2 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{x+11} = x + 5$.

Square both sides: $x + 11 = (x + 5)^2$.

$x^2 + 9x + 14 = 0$, so $x = -7$ or $x = -2$.

Check in $\sqrt{x+11} = x + 5$: $x = -2$ gives $3 \geq 0$ (valid).

$x = -7$ gives right-hand side $-2 < 0$, so it is rejected.

Answer: $x = -2$

3 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_3(x + 1) + \log_3(x - 1) = 1$,
justifying clearly any root that you reject.

$(x + 1)(x - 1) = 3^1 = 3$

$x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$, so $x = 2$ or $x = -2$

$x = -2$ makes $x + 1 = -1 < 0$ and log of a negative number is undefined

Answer: $x = 2$

4 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $5k = n^2 + 3$ has no integer solutions (k and n integers).

Every integer is $5q$, $5q + 1$, $5q + 2$, $5q + 3$, $5q + 4$ for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 5$, so:

$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 4$; $4 \to 1$ ($r \to n^2 \bmod 5$)

So $n^2 + 3 \equiv 3$ or 4 or 2 (mod 5).

No residue is 0, so $n^2 + 3$ is never a multiple of 5.

Answer: Proved

5 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 25 observations x, $\Sigma(x - 60) = 45$ and $\Sigma(x - 60)^2 = 381$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 60) = \frac{45}{25} = 1.8$, so mean of $x = 61.8$

Variance $= \frac{381}{25} - (1.8)^2 = 12$

$\sigma = \sqrt{12}$

$\Sigma x^2 = \Sigma(x - 60)^2 + 120\,\Sigma(x - 60) + 25 \times 60^2$

$= 381 + 5400 + 90000$

Answer: (a) 61.8 (b) 3.46 (c) 95781

6 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_1^e \frac{1}{x(\ln x + 3)}\,dx$$

Let $u = \ln x + 3$, $du = \frac{1}{x}dx$

Integral $= \int_3^4 \frac{1}{u}\,du = [\ln u]$

$= \ln 4 - \ln 3$

Answer: $2\ln 2 - \ln 3$

7 Normal approximation to a Poisson (CIE)

The number of customers arriving at a shop in one hour is Po(36).
Use a suitable approximation to find P(X ≥ 42), to 3 significant figures.

$\lambda = 36$ is large (greater than 15), so X is approximately N(36, 36)
Standard deviation $= \sqrt{36} = 6.000$
Continuity correction:
$P(X \geq 42) \approx P(Y > 41.5) = P(Z > 0.917) = 0.180$
Answer: 0.180

8 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 140 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{3}{4}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{12}{5}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 140\cos\alpha$, so $R = 140 \times \frac{4}{5} - P\sin\beta$

Along the plane: $P\cos\beta = 140\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 140(\sin\alpha + \mu\cos\alpha)$

(a) $P(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13}) = 140(\frac{3}{5} + \frac{3}{4} \times \frac{4}{5})$, so $P = 156$

(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)

$P_{min} = 140(\frac{3}{5} + \frac{3}{4} \times \frac{4}{5}) \div \frac{5}{4}$

Answer: (a) 156 N (b) 134.4 N

9 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−2, −3, −3), (1, −2, 4), (1, −2, −3).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} -3 \\ -1 \\ -7 \end{pmatrix}$, $BC = \begin{pmatrix} 0 \\ 0 \\ -7 \end{pmatrix}$; $BA \cdot BC = 49$

$|BA| = \sqrt{59}$, $|BC| = \sqrt{49}$

$\cos ABC = \frac{49}{\sqrt{59}\sqrt{49}} = 0.9113$, so $ABC = 24.3°$

$|BA \times BC| = \left|\begin{pmatrix} 7 \\ -21 \\ 0 \end{pmatrix}\right| = \sqrt{490}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 24.3°, area $= \frac{7}{2}\sqrt{10}$

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10 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $5^{2x} - 130 \times 5^x + 625 = 0$.

Let $y = 5^x$. Then $y^2 - 130y + 625 = 0$.
$(y - 5)(y - 125) = 0$, so $y = 5$ or $y = 125$.
$5^x = 5$ gives $x = 1$; $5^x = 125$ gives $x = 3$.
Answer: $x = 1$ or $x = 3$