Day 151 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = x^2 - 2x + 3$ for $x \geq k$. Given that k is the least value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = (x - 1)^2 + 2$.
The vertex is at x = 1, so k = 1 (f is one-one on one side of it).
Let y = f(x): $y - 2 = (x - 1)^2$, so $(x - 1) = \pm\sqrt{y - 2}$.
As $x \geq 1$, take the positive root: $x = 1 + \sqrt{y - 2}$.
Range of f = domain of f^{-1}: f(1) = 2, so $x \geq 2$.
Answer: k = 1; $f^{-1}(x) = 1 + \sqrt{x - 2}$; $x \geq 2$

2 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_5 x + \log_5 y = 3$
$x - y = 20$

$\log_5(xy) = 3$, so $xy = 5^3 = 125$
$x = y + 20$: $y^2 + 20y - 125 = 0$
$(y - 5)(y + 25) = 0$; $y > 0$ so $y = 5$
Then $x = y + 20$
Answer: x = 25, y = 5

3 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The daily output of a machine is N(μ, 9) with known variance 9. The mean was claimed to be 86 units. A random sample of 16 has mean 84. Test at the 10% level whether the mean is greater than 86.

H_0: μ = 86, H_1: μ > 86. Under H_0, the sample mean ~ $N(86, \frac{9}{16})$
$z = \frac{84 - 86}{3 \div \sqrt{16}} = -2.667$
p-value = $P(Z > -2.667) = 0.996$
0.996 > 10%, so do not reject H_0
There is insufficient evidence at the 10% level that the mean has changed as claimed
Answer: Do not reject H_0 ($z = -2.667$, p-value 0.996)

4 Constant speed up a slope, force at an angle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 195 N is pulled at constant speed up a line of greatest slope of a rough plane ($\tan\alpha = \frac{5}{12}$, $\mu = \frac{3}{4}$) by a force P N at angle β above the plane.

(a) Find P when $\tan\beta = \frac{12}{5}$. (b) Find the least possible value of P as β varies.

Perpendicular to the plane: $R + P\sin\beta = 195\cos\alpha$, so $R = 195 \times \frac{12}{13} - P\sin\beta$
Along the plane: $P\cos\beta = 195\sin\alpha + \mu R$, so $P(\cos\beta + \mu\sin\beta) = 195(\sin\alpha + \mu\cos\alpha)$
(a) $P(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13}) = 195(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13})$, so P = 195
(b) P is least when $\cos\beta + \mu\sin\beta$ is greatest, i.e. $\sqrt{(1 + \mu^2)} = \frac{5}{4}$ (when $\tan\beta = \frac{3}{4}$)
$P_{min} = 195(\frac{5}{13} + \frac{3}{4} \times \frac{12}{13}) \div \frac{5}{4}$
Answer: (a) 195 N (b) 168 N

5 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.
Initially P = 80 and after 4 hours P = 160.
Find k, and the time for the population to reach 400, to 3 s.f.

$P_0 = 80$. $80\,e^{4k} = 160$ so $e^{4k} = 2$

$k = \frac{1}{4}\ln 2 = 0.173$

$80\,e^{kt} = 400$ so $e^{kt} = 5$, $t = \frac{1}{k}\ln 5$

t = 9.29 hours

Answer: k = 0.173, t = 9.29 hours

6 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at $x = -2$ and touches at x = 4, so $f(x) = k(x + 2)(x - 4)^2$.
At (0, 32): $32 = k(2)(-4)^2 = 32k$, so k = 1.
$f(x) = (x + 2)(x - 4)^2$, then expand.
Answer: $f(x) = x^3 - 6x^2 + 32$

7 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = ln x to find the exact value of

$$\int_1^{e^2} \frac{(\ln x)^3}{x}\,dx$$

$du = \frac{1}{x}dx$; limits x = 1, e^2 become u = 0, 2

Integral = $\int_0^2 u^3\,du$
Evaluate the limits
Answer: 4

8 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{-11x + 3}{(1 - x)(1 - 3x)}$.
(a) Express f(x) in partial fractions.
(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $-11x + 3 \equiv A(1 - 3x) + B(1 - x)$: A = 4, B = −1.
$f(x) = \frac{4}{1 - x} - \frac{1}{1 - 3x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use c = −1, then c = −3).
Coefficients of x^j: $4(1)^j - 1(3)^j$.
Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) A = 4, B = −1 (b) $3 + x - 5x^2 - 23x^3 + \ldots$; $|x| < \frac{1}{3}$

9 Conditional probability with a normal variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bulb's lifetime, in hours, is $N(64, 10^2)$.
Given that it exceeds 64, find the probability that it exceeds 70.

Required = $P(X > 70 \mid X > 64) = \frac{P(X > 70)}{P(X > 64)}$
P(X > 70) = P(Z > 0.600) = 0.2743
P(X > 64) = P(Z > 0.000) = 0.5000
$= \frac{0.2743}{0.5000} = 0.549$
Answer: 0.549

10 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car P passes a point O travelling at a constant speed of 25 m s^{-1}. 4 s later, car Q leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.
Q overtakes P at the point X. Find (a) the time Q takes to reach X, (b) the distance OX, (c) the speed of Q at X.

Let Q travel for t s. P: distance = 25(t + 4); Q: distance = ½ × 3 × t^2
Equate: $3t^2 - 50t - 200 = 0$, so $(t - 20)(3t + 10) = 0$
t = 20 (the other root is negative)
OX = 3 × 20^2 / 2 = 600 m, and speed = 3 × 20 = 60
Answer: (a) 20 s (b) 600 m (c) 60 m s^{-1}