Day 147 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from level ground with horizontal and vertical velocity components 19.6 and 44.1 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 19.6$
Moving upwards: $44.1 - 9.8t = 19.6$, so $t = 2.5$
Moving downwards: $44.1 - 9.8t = -19.6$, so $t = 6.5$
Horizontal distance = $19.6 \times (6.5 - 2.5)$
Answer: (a) 2.5 s and 6.5 s (b) 78.4 m

2 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{12}{x^2 + 2x + 3}$ for all real x.
Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 + 2x + 3 = (x + 1)^2 + 2$.
$(x + 1)^2 \geq 0$, so the denominator is at least 2, with equality at $x = -1$.
The denominator is always positive, so $f(x) > 0$.
Greatest $f = \frac{12}{2} = 6$ at $x = -1$.
Answer: greatest value 6 at $x = -1$; range $0 < f(x) \leq 6$

3 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 4x + 16$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(4x + 16)}$.
(b) Use $x_{n+1} = \sqrt[3]{(4x_n + 16)}$ with $x_0 = 3$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(4x + 16)}$
$x_1 = 3.0366$, $x_2 = 3.0419$, $x_3 = 3.0426$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 3.043$
Answer: $x_1 = 3.0366$, $x_2 = 3.0419$, $x_3 = 3.0426$; $\alpha = 3.043$

4 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 - x - 20 \leq 0$ and $-2x - 5 < 3$.

Quadratic: $(x + 4)(x - 5) \leq 0$ gives $-4 \leq x \leq 5$.
Linear: $-2x - 5 < 3$ gives $x > -4$. (the inequality reverses as we divide by a negative)
Both must hold: take the overlap of the two sets.
Answer: $-4 < x \leq 5$

5 Projectile from a cliff (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from the top of a vertical cliff, 19.6 m high, at 24.5 m s^{-1} at angle α above the horizontal ($\tan \alpha = \frac{3}{4}$), and lands on horizontal ground.
Find (a) the time of flight and the distance from the cliff, (b) the speed and direction on landing. Take g = 9.8 m s^{-2}.

Vertically (up positive): $-19.6 = 14.7t - 4.9t^2$, so $4.9t^2 - 14.7t - 19.6 = 0$
$t = 4$ s (the positive root)
Horizontal distance = $19.6 \times 4 = 78.4$ m
On landing: $v_x = 19.6$, $v_y = 14.7 - 9.8 \times 4 = -24.5$
speed = $\sqrt{(19.6^2 + 24.5^2)} = 31.4$, $\tan \text{angle} = \frac{24.5}{19.6}$, angle = 51.3° below the horizontal
Answer: (a) 4 s, 78.4 m (b) 31.4 m s^{-1} at 51.3° below the horizontal

6 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$P(X = x) = kx(6 - x)$ for $x = 1, 2, ..., 5$, and 0 otherwise.

(a) Show that $k = \frac{1}{35}$.

(b) Find E(X) and Var(X).

(c) Find Var(4X + 5).

(a) $\Sigma P(X = x) = k(5 + 8 + 9 + 8 + 5) = 35k = 1$, so $k = \frac{1}{35}$

(b) $E(X) = \Sigma x P(X = x) = \frac{1\times5 + 2\times8 + 3\times9 + 4\times8 + 5\times5}{35} = 3$

$E(X^2) = \frac{1\times5 + 4\times8 + 9\times9 + 16\times8 + 25\times5}{35} = \frac{53}{5}$

$Var(X) = \frac{53}{5} - 3^2 = \frac{8}{5}$

(c) $Var(4X + 5) = 4^2 Var(X) = 16 \times \frac{8}{5} = \frac{128}{5}$

Answer: (b) E(X) = 3, Var(X) = $\frac{8}{5}$ (c) $\frac{128}{5}$

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2\sqrt{180 - x}$, for $0 < x < 180$, has one stationary point. Find its exact coordinates and determine its nature.

$dy/dx = \frac{2x(720 - 5x)}{2\sqrt{180 - x}}$

$dy/dx = 0$: $720 - 5x = 0$, so $x = 144$

$y = 2 \times 144^2 \times 6$

Gradient changes from positive to negative: a maximum

Answer: (144, 248832), maximum

8 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 64)$. H_0: $\mu = 52$ is tested against H_1: $\mu > 52$ using the mean of a sample of size 25 at the 5% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when $\mu = 54.5$.

Under H_0, sample mean $\sim N(52, \frac{64}{25})$, standard error = 1.600

z = 1.6449 for the upper tail: critical value = 52 + 1.6449 × 1.600 = 54.63

Type II error: do not reject H_0, sample mean < 54.63, where mean ~ N(54.5, 1.600)

$z = \frac{54.63 - 54.5}{1.600} = 0.082$

Probability = 0.533

Answer: (a) sample mean > 54.6 (b) 0.533

9 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each year Jo pays £250 into an account on 1 January.

Interest of 3% is added each 31 December.

Find the first year-end at which the balance exceeds £15,000.

After n years: $250(1.03) + 250(1.03^2) + ... + 250(1.03^n)$.

GP: $a = 250 \times 1.03$, $r = 1.03$: $S_n = \frac{250 \times 1.03\,(1.03^n - 1)}{0.03}$.

$S_n > 15000 \Rightarrow 1.03^n > 2.7476 \Rightarrow n > 34.193$.

So n = 35 years.

Answer: after 35 years

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix}-2\\3\\-2\end{pmatrix} + t\begin{pmatrix}1\\1\\-3\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-7\\5\\-1\end{pmatrix} + t\begin{pmatrix}4\\-1\\-1\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-5\\2\\1\end{pmatrix} + t\begin{pmatrix}3\\-2\\2\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}3\\-2\\2\end{pmatrix} = 0$

$-17 + t(17) = 0$, so $t = 1$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-2\\0\\3\end{pmatrix}$, with length $\sqrt{13}$

Answer: t = 1, minimum distance = $\sqrt{13}$ m