Day 146 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 3 and its common difference d is non-zero. The 2nd, 6th and 14th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: $3 + d$, $3 + 5d$, $3 + 13d$.

Geometric: $(3 + 5d)^2 = (3 + d)(3 + 13d)$.

Expand: $25d^2 + 30d + 9 = 13d^2 + 42d + 9$, so $d(12d - 12) = 0$.

$d \neq 0$, so $d = 1$. Ratio $= \frac{8}{4} = 2$.

Answer: d = 1, ratio = 2

2 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 12 s, P is back at its starting point.

Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 3 \times 6 + \frac{1}{2}(6 - W)(4) - W(5) = 0$

$9 + 12 - 2W - 5W = 0$, so $7W = 21$

$W = 3$

$v = 0$ where the line from (3, 6) to (7, −3) crosses the axis: $t = 3 + 4 \times \frac{6}{9} = \frac{17}{3}$

Greatest distance = area of triangle $= \frac{1}{2} \times \frac{17}{3} \times 6$

Answer: (a) W = 3 (b) 17 m

3 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 50 observations x, $\Sigma(x - 40) = 30$ and $\Sigma(x - 40)^2 = 768$.

(a) Find the mean of x.

(b) Find the standard deviation of x.

(c) Find the value of Σx^2.

Mean of $(x - 40) = \frac{30}{50} = 0.6$, so mean of $x = 40.6$

Variance $= \frac{768}{50} - (0.6)^2 = 15$

$\sigma = \sqrt{15}$

$\Sigma x^2 = \Sigma(x - 40)^2 + 80\,\Sigma(x - 40) + 50 \times 40^2$

$= 768 + 2400 + 80000$

Answer: (a) 40.6 (b) 3.87 (c) 83168

4 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^3 + 3x^2 - 4x + 6}{(x + 2)(x - 1)}$ in the form $ax + b + \frac{C}{(x + 2)} + \frac{D}{(x - 1)}$.

Divide: the numerator has degree 3, the denominator degree 2.

$(x^3 + 3x^2 - 4x + 6) \div (x^2 + x - 2)$ gives quotient $x + 2$.

Remainder: $-4x + 10$.

Put $x = -2$: remainder $= C(-3)$ so $C = -6$. Put $x = 1$: $D = 2$.

Answer: $x + 2 - \frac{6}{(x + 2)} + \frac{2}{(x - 1)}$

5 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x e^y + y = 2$.
Find the equation of the tangent to C at the point (2, 0),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (2, 0): $dy/dx = -\frac{1}{3}$

$y - 0 = (-\frac{1}{3})(x - 2)$, then clear fractions

Answer: $x + 3y - 2 = 0$

6 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 10 pairs of values: $\Sigma x = 110$, $\Sigma y = 350$, $\Sigma x^2 = 1406$, $\Sigma xy = 3948$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 7 and 16. Estimate y when x = 15 and comment on its reliability.

$S_{xx} = 1406 - \frac{110^2}{10} = 196$

$S_{xy} = 3948 - \frac{110 \times 350}{10} = 98$

$b = \frac{S_{xy}}{S_{xx}} = \frac{98}{196} = \frac{1}{2}$

$a = \bar{y} - b\,\bar{x} = 35 - (\frac{1}{2})(11) = 29.5$

$y = 29.5 + 0.5x$
At x = 15: y = 37. This is extrapolation (outside the data range), so may be unreliable
Answer: (a) $y = 29.5 + 0.5x$ (b) 37, unreliable (extrapolation)

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = 3\cos x$ and $y = 3\cos 2x$ meet at x = 0 and
at $x = \frac{2\pi}{3}$. Find the exact area enclosed between them
for $0 \le x \le \frac{2\pi}{3}$.

For $0 < x < \frac{2\pi}{3}$, $\cos x > \cos 2x$

Area $= 3\int_0^{2\pi/3} (\cos x - \cos 2x)\,dx = 3[\sin x - \frac{\sin 2x}{2}]$

$= 3[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}]$

Answer: $\frac{9}{4}\sqrt{3}$

8 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + ax^2 + bx + 12$, where a and b are constants.
$(x - 3)$ is a factor of f(x), and f(x) leaves remainder 6 when divided by $(x - 1)$.
Find a and b, and hence solve f(x) = 0.

f(3) = 0: $27 + 9a + 3b + 12 = 0$, so $9a + 3b + 39 = 0$.
f(1) = 6: $a + b + 13 = 6$.
Solving: $a = -3$, $b = -4$.
$f(x) = x^3 - 3x^2 - 4x + 12 = (x + 2)(x - 2)(x - 3)$
Answer: $a = -3$, $b = -4$; $x = -2, 2, 3$

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 5 kg lies on a smooth horizontal table, with a block of mass 5 kg on it ($\mu = 0.25$ between them). A horizontal force P N acts on the block.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 36.75. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{10}$. The plank is accelerated only by friction: $F = 5a$

Limiting friction: $F = \mu mg = 0.25 \times 5 \times 9.8 = 12.25$, so $a = \frac{12.25}{5} = 2.45$

P_max $= 10 \times 2.45 = 24.5$

$P = 36.75 > 24.5$, so the block slips. Plank: $a = \frac{12.25}{5} = 2.45$

Block: $36.75 - 12.25 = 5a$, so $a = 4.9$

Answer: (a) 24.5 N (b) block 4.9 m s^{-2}, plank 2.45 m s^{-2}

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} -9 \\ 7 \\ 7 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0$

$-15 + t(5) = 0$, so $t = 3$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$, with length $\sqrt{29}$

Answer: t = 3, minimum distance = $\sqrt{29}$ m