Day 145 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 15 pairs of values: $\Sigma x = 150$, $\Sigma y = 225$, $\Sigma x^2 = 1643$, $\Sigma xy = 1964$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 6 and 15. Estimate y when x = 18 and comment on its reliability.

$S_{xx} = 1643 - \frac{150^2}{15} = 143$

$S_{xy} = 1964 - \frac{150 \times 225}{15} = -286$

$b = \frac{S_{xy}}{S_{xx}} = \frac{-286}{143} = -2$

$a = \bar{y} - b\,\bar{x} = 15 - (-2)(10) = 35$

$y = 35 - 2x$

At x = 18: y = −1. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) $y = 35 - 2x$ (b) −1, unreliable (extrapolation)

2 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 9 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 2 \times 8 + \frac{1}{2}(8 - W)(4) - W(3) = 0$

$8 + 16 - 2W - 3W = 0$, so $5W = 24$

$W = 4.8$

v = 0 where the line from (2, 8) to (6, −4.8) crosses the axis: $t = 2 + 4 \times \frac{8}{12.8} = 4.5$

Greatest distance = area of triangle = $\frac{1}{2} \times 4.5 \times 8$

Answer: (a) W = 4.8 (b) 18 m

3 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $-x^4 + 20x^2 - 64 \geq 0$.

Use the factor theorem to find roots: −4, −2, 2, 4.

$f(x) = -(x + 4)(x + 2)(x - 2)(x - 4)$.

Sign of f in each region (left to right): −, +, −, +, −.

Take the regions where $f \geq 0$ (include roots if ≤ or ≥).

Answer: $-4 \leq x \leq -2$ or $2 \leq x \leq 4$

4 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves with constant acceleration. At t = 0 seconds P is at $(-\mathbf{i} + \mathbf{j})$ m with velocity $(10\mathbf{i} + 13\mathbf{j})$ m s^{-1}. At t = 10, P has velocity $(-10\mathbf{i} - 7\mathbf{j})$ m s^{-1}.
Find (a) the acceleration of P, (b) the time at which P moves parallel to $(\mathbf{j})$, (c) the position vector of P at that time.

$\mathbf{a} = \frac{\mathbf{v} - \mathbf{u}}{t} = \frac{(-10\mathbf{i} - 7\mathbf{j}) - (10\mathbf{i} + 13\mathbf{j})}{10} = (-2\mathbf{i} - 2\mathbf{j})$

$\mathbf{v} = (10\mathbf{i} + 13\mathbf{j}) + t(-2\mathbf{i} - 2\mathbf{j})$

Parallel to $(\mathbf{j})$: $(-2t + 10) \times 1 = (-2t + 13) \times 0$, giving t = 5

$\mathbf{r} = (-\mathbf{i} + \mathbf{j}) + 5(10\mathbf{i} + 13\mathbf{j}) + \frac{1}{2} \times 5^2\,(-2\mathbf{i} - 2\mathbf{j}) = (24\mathbf{i} + 41\mathbf{j})$

Answer: (a) $(-2\mathbf{i} - 2\mathbf{j})$ m s^{-2} (b) t = 5 (c) $(24\mathbf{i} + 41\mathbf{j})$ m

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 18 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 100$, and after 8 minutes $\theta = 59$. Find A and k, then find when $\theta = 36$ (3 s.f.).

$t = 0$: $18 + A = 100$ so $A = 82$

$t = 8$: $18 + 82e^{-8k} = 59$ so $e^{-8k} = \frac{1}{2}$

$k = \frac{1}{8} \ln 2 = 0.0866$

$82\,e^{-kt} = 18$ so $t = \frac{1}{k} \ln \frac{41}{9}$

$t = 17.5$ minutes

Answer: A = 82, k = 0.0866, t = 17.5 minutes

6 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 7$ and $v = \frac{y}{10}$. The regression line of v on u is $v = 37 + 1u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 15$.

Substitute $u = x - 7$ and $v = y/10$: $\frac{y}{10} = 37 + 1(x - 7)$

$y = 10[37 + 1x - 7] = 10 \times 30 + 10 \times 1x$

$y = 300 + 10x$

At $x = 15$: $y = 300 + 10 \times 15 = 450$

Answer: (a) y = 300 + 10x (b) 450

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For $0 \le x \le \pi$, the curves $y = 6 \sin x$ and $y = 6 \sin 2x$ meet at the origin and at $x = \frac{\pi}{3}$. Find the exact area enclosed between them for $0 \le x \le \frac{\pi}{3}$.

For $0 < x < \frac{\pi}{3}$, $\sin 2x > \sin x$

Area $= 6\int_0^{\pi/3} (\sin 2x - \sin x)\,dx = 6[-\frac{\cos 2x}{2} + \cos x]$

$= 6[(-\cos\frac{2\pi}{3}/2 + \cos\frac{\pi}{3}) - (-\frac{1}{2} + 1)] = 6[\frac{3}{4} - \frac{1}{2}]$

Answer: $\frac{3}{2}$

8 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{1}{2}u_n + 6$, $u_1 = -6$.

(a) Find the limit L of the sequence.
(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = \frac{1}{2}L + 6$, so $L(1 - \frac{1}{2}) = 6$.

$L = 12$.

(b) $u_n - L = (u_1 - L)(\frac{1}{2})^{n-1} = (-18)(\frac{1}{2})^{n-1}$.

Need $18 \times (\frac{1}{2})^{n-1} < 0.001$, so $n - 1 > 14.136$.

The least n is 16.

Answer: (a) L = 12 (b) n = 16

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9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 2xy + 4y^2 = 48$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -2y}{-2x + 8y}$

Horizontal tangent when $2x + -2y = 0$, so $y = \frac{2x}{2}$

Substitute into the equation of C: $x^2 + (-2xy) + (4y^2) = 48$
Solve for the remaining coordinate
Answer: $(-4, -4)$ and $(4, 4)$

10 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each year Jo pays £300 into an account on 1 January.
Interest of 6% is added each 31 December.
Find the first year-end at which the balance exceeds £12,000.

After n years: $300(1.06) + 300(1.06^2) + ... + 300(1.06^n)$.

GP: $a = 300 \times 1.06$, $r = 1.06$: $S_n = \frac{300 \times 1.06\ (1.06^n - 1)}{0.06}$.

$S_n > 12000 \Rightarrow 1.06^n > 3.2642 \Rightarrow n > 20.302$.
So n = 21 years.
Answer: after 21 years