Day 144 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{8}{x^2 + 4x + 8}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 + 4x + 8 = (x + 2)^2 + 4$.

$(x + 2)^2 \geq 0$, so the denominator is at least 4, with equality at $x = -2$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{8}{4} = 2$ at $x = -2$.

Answer: greatest value 2 at $x = -2$; range $0 < f(x) \leq 2$

2 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(22, \frac{2}{3})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{22 - r}{r + 1} \times 2$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(22, r + 1)}{C(22, r)} \times \frac{p}{q} = \frac{22 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{2}{3}$ and $q = \frac{1}{3}$, $p \div q = 2$

The ratio exceeds 1 when $(22 - r)\frac{2}{3} > (r + 1)\frac{1}{3}$, that is $r < \frac{43}{3}$

So probabilities rise up to r = 15 and then fall; the mode is 15

$P(X = 15) = 0.178$

Answer: 15 with probability 0.178

3 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $7k = n^2 + 4$ has no integer solutions (k and n integers).

Consider $n = 7q + r$, with $r = 0, 1, ..., 6$.

Squaring, $n^2 \equiv r^2 \pmod 7$, so:

$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 2$; $4 \to 2$; $5 \to 4$; $6 \to 1$ ($r \to n^2 \bmod 7$)

So $n^2 + 4 \equiv 4$ or 5 or 6 or 1 (mod 7).

No residue is 0, so $n^2 + 4$ is never a multiple of 7.

Answer: Proved

4 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 100 m wide and flows at 3 m s^{-1}. A boatman can row at 5 m s^{-1} in still water and wants to reach the point directly opposite his starting point.

Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current

Let the boat head at angle φ upstream from the perpendicular: $5 \sin \varphi = 3$, so $\varphi = 36.9°$

Angle to the bank = $90° - \varphi = 53.1°$

Speed across = $\sqrt{(5^2 - 3^2)} = 4$ m s^{-1}

Time = $\frac{100}{4} = 25.0$ s

Answer: (a) 53.1° (b) 25.0 s

5 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\cos A = \frac{5}{13}$ and $\sin B = \frac{12}{13}$.

Find the exact values of $\tan(A + B)$ and $\tan(A - B)$.

$\cos A = \frac{5}{13}$, $\sin A = \frac{12}{13}$; $\sin B = \frac{12}{13}$, $\cos B = \frac{5}{13}$

$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\frac{120}{169}}{-\frac{119}{169}} = -\frac{120}{119}$

$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{0}{1} = 0$

Answer: $\tan(A + B) = -\frac{120}{119}$, $\tan(A - B) = 0$

6 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(8 + x)^{\frac{1}{3}}$ up to and including the term in x^2.

(b) Use $x = 0.04$ to estimate $\sqrt[3]{8.04}$ to 5 decimal places.

(a) $(8 + x)^{\frac{1}{3}} = 2(1 + \frac{1}{8}x)^{\frac{1}{3}}$.

$= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + ...$

(b) $8 + x$ with $x = 0.04$ gives 8.04.

Substitute: $2 + \frac{1}{12}(0.04) + (-\frac{1}{288})(0.04)^2$.

= 2.00333 (5 d.p.).

Answer: (a) $2 + \frac{1}{12}x - \frac{1}{288}x^2$ (b) 2.00333

7 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $4\text{cosec}^2\theta + 7\cot\theta - 6 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $4\cot^2\theta + 7\cot\theta - 2 = 0$
$(\cot\theta + 2)(4\cot\theta - 1) = 0$

$\cot\theta = -2$ or $\cot\theta = \frac{1}{4}$

$\tan\theta = -\frac{1}{2}$ or 4, so $\theta = 153.4°$ and $76.0°$ (+180°)

Answer: $\theta = 76.0°, 153.4°, 256.0°, 333.4°$

8 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (2 kg) and B (10 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\alpha = 30°$, and B is 4 m above the ground.

Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 2 \times 9.8 \times \frac{1}{2} = 2a$. B: $10 \times 9.8 - T = 10a$

Adding: $9.8(10 - 2 \times \frac{1}{2}) = 12a$, so $a = 7.35$

$T = 10(9.8 - 7.35) = 24.5$

When B lands: $v^2 = 2 \times 7.35 \times 4 = 58.8$

Then A decelerates at $g \sin\alpha = 4.9$: further distance $= \frac{58.8}{2 \times 4.9} = 6$

Answer: 6 m

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9 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The mass of a parcel is N(81, 3^2) kg. A parcel is 'heavy' if it exceeds 85.
(a) Find the probability that a parcel is heavy.
(b) 10 parcels are chosen at random. Find the probability that at least 2 are heavy.

(a) P(X > 85) = P(Z > 1.333) = 0.09121
(b) Let Y = number heavy, Y ~ B(10, 0.09121)
P(Y ≥ 2) = 1 − P(Y ≤ 1) = 0.230
Answer: (a) 0.0912 (b) 0.230

10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Write $\frac{2x + 13}{(x + 5)^2}$ in the form $\frac{A}{x + 5} + \frac{B}{(x + 5)^2}$.

(b) Find the exact value of $\int_0^3 \frac{2x + 13}{(x + 5)^2}\,dx$.

$2x + 13 \equiv A(x + 5) + B$, so A = 2, and B = 3

Integral = $[2\ln(x + 5) - \frac{3}{x + 5}]$

Substitute limits 0 and 3

Answer: (a) A = 2, B = 3; (b) $\frac{9}{40} + 6\ln 2 - 2\ln 5$